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Reg. No. : ....

Name : ....

Second Semester M.Sc. Degree Examination, July 2019

Mathematics

MM 221 : ABSTRACT ALGEBRA

(2017 Admission onwards)

Time : 3 Hours Max. Marks : 75

Answer **five** questions choosing Part – A or Part – B from each question.

All questions carry equal marks.

1. (A) (a) Let G be a group and $a \in G$. If a is infinite order, prove that $a^i = a^j$ if and only if $i = j$. If a is of finite order n, $a^i = a^j$ if and only if n divides $i - j$. **5**

(b) State and prove Lagrange's theorem. **6**

(c) Prove that the set of even permutations in S_n forms a normal subgroup of S_n. **4**

OR

(B) (a) Show that every permutation can be written as a cycle or as a product of disjoint cycles. **6**

(b) State and prove Cayley's theorem. **6**

(c) Find the automorphism group $Aut\ (\mathbb{Z}_{10})$. **3**

P.T.O.

2. (A) (a) State and prove the Class equation for a finite group. 5

(b) Let p be a prime and let G be a group of order $p^k m$, where p does not divide m. Show that the number n of Sylow p-subgroups of G is equal to 1 modulo p and divides m. Hence or otherwise prove that, any two Sylow p-subgroups of G are conjugates. 6

(c) Show that the only group of order 255 is $\mathbb{Z}_{255}$. 4

OR

(B) (a) Let G be a finite group and let p be a prime. If p^k divides $|G|$, show that G has atleast one subgroup of order p^k. 5

(b) Show that any group G of order 72 must have a proper, nontrivial normal subgroup. 5

(c) Show that A_5 is simple. 5

3. (A) (a) Show that the ideal $\langle x^2+1 \rangle$ is maximal is $\mathbb{R}[x]$. 5

(b) Let D be an integral domain. Show that there exists a field F that contains a subring isomorphic to D. 10

OR

(B) (a) Let R be a commutative ring with unity and let A be an ideal of R. Prove that R/A is an integral domain if and only if A is prime. 5

(b) Prove that the product of two primitive polynomials is primitive. 6

(c) State and prove the Eisensteins Criterion. 4

4. (A) (a) Show that $\mathbb{Z}[\sqrt{-3}]$ contains an element which is irreducible but not prime. **4**

(b) Let F be a field and let $f(x)$ be a nonconstant element of $F[x]$. Show that there exists a splitting field E for $f(x)$ over F. **5**

(c) Let F be a field, let $p(x) \in F[x]$ be irreducible over F, and let a be a zero of $p(x)$ in some extension of F. If f is a field isomorphism from F to F' and b is a zero of $f(p(x))$ in some extension of F', prove that there is an isomorphism from $F(a)$ to $F'(b)$ that agrees with f on F and carries a to b. **6**

OR

(B) (a) Prove that every principal ideal domain is a unique factorization domain. **6**

(b) Let F be a field and let $p(x) \in F[x]$ be irreducible over F. If a is a zero of $p(x)$ in some extension E of F, prove that $F(a)$ is isomorphic to $F[x]/\langle p(x) \rangle$. **4**

(c) Let $f(x)$ be an irreducible polynomial over a field F and let E be a splitting field of $f(x)$ over F. Show that all the zeros of $f(x)$ in E have the same multiplicity. **5**

5. (A) (a) If F is a field of characteristic 0, and a and b are algebraic over F, prove that there is an element c in $F(a, b)$ such that $F(a, b) = F(c)$. **6**

(b) Prove that a factor group of a solvable group is solvable. **4**

(c) Prove that $Gal(GF(p^n)/GF(p)) \approx \mathbb{Z}_n$. **5**

OR

(B) (a) If K is an algebraic extension of E and E is an algebraic extension of F, show that K is an algebraic extension of F. **10**

(b) Draw the diagram for lattice of subfields of $\mathbb{Q}(\sqrt[4]{2}, i)$. **5**
