(Pages : 4) **L – 6307**

Reg. No. : ....

Name : ....

First Semester M.Sc. Degree Examination, August 2021

Mathematics

MM 211 : LINEAR ALGEBRA

(2020 Admission)

Time : 3 Hours Max. Marks : 75

PART – A

Answer any **five** questions from among the questions 1 to 8. Each question carries **3** marks.

1. Find a number *t* such that (3, 1, 4), (2, –3, 5), (5, 9, *t*) is not linearly independent in $\mathbb{R}$.

2. Give an example for an infinite dimensional vector space. Prove the necessary details.

3. Give an example of a linear map T such that dim *null* $T = 3$ and dim *range*$T = 2$.

4. Suppose $T:\mathbb{R}^2 \to \mathbb{R}^3$ is defined by $T(x, y) = (2x, 5y, x+y)$. Find *a* basis for the range T.

5. Define $T:\mathbb{F}^3 \to \mathbb{F}^3$ by $T(z_1, z_2, z_3) = (3z_2, 0, z_3)$. Find all eigen values and eigen vectors of T.

6. Suppose $P \in \mathcal{L}(V)$ and $P^2 = P$. Prove that $V = null\ P \oplus range\ P$.

P.T.O.

7. Suppose $T \in L(\mathbb{C}^3)$ is defined by $T(z, z_2, z_3) = (z_2, z_3, 0)$. Prove that T has no square root.

8. Prove or give counter example. A matrix is invertible if and only if its trace in nonzero.

(5 × 3 = 15 Marks)

PART – B

Answer **all** questions from 9 to 13. **Each** question carries **12** marks.

9. (a) (i) Suppose U and W are subspaces of V. Then show that $U + W$ is a direct sum if and only if $U \cap W = \{0\}$.

(ii) Suppose $U = \{(x, y, x+y, x-y, 2x) \in F^5 ; x, y \in F\}$ Find three subspaces W_1, W_2, W_3 of F^5 none of which equals $\{0\}$, such that $F^5 = U \oplus W_1 \oplus W_2 \oplus W_3$.

(iii) Show that the subspace of $\mathbb{R}^3$ are precisely $\{0\}$, $\mathbb{R}^3$, all lines $\mathbb{R}^3$ through the origin, and all planes in $\mathbb{R}^3$ through the origin.

OR

(b) (i) Show that every linearly independent list of vectors in a finite-dimensional vector space can be extended to a basis of the vector space.

(ii) Suppose v_1, v_2, v_3, v_4 is linearly independent in V. Prove that the list $v_1 - v_2$, $v_2 - v_3$, $v_3 - v_4$, v_4 is also linearly independent.

(iii) Suppose U_1 and U_2 are subspaces of V. Prove that the intersection $U_1 \cap U_2$ is a subspace of V. Derive the condition under which $U_1 \cup U_2$ is a subspace.

10. (a) (i) Let V be a finite dimensional vector space. Does there exist a linear transformation $T : V \to V$ such that T is injective but not surjective.

(ii) Suppose $D : P_5(\mathbb{R}) \to P_4(\mathbb{R})$ is the differentiation map defined by $Dp = p'$. Find the matrix of D with respect to the standard bases of $P_5(\mathbb{R})$ and $P_4(\mathbb{R})$

OR

(b) (i) For a linear transformation T between two finite dimensional vector spaces V, W, introduce the matrix of T with respect to some given bases. What is the matrix of the composition S T of two linear transformations S, T defined on appropriate spaces?

(ii) Suppose V is finite-dimensional and $T \in \mathcal{L}(V, W)$. Then range T is finite-dimensional and $\dim V + \dim \textit{null } T = \dim \textit{range } T$.

11. (a) (i) Let $T : V \to V$ be a linear map. Suppose $\lambda_1, \lambda_2, ...\lambda_m$ are distinct eigen values of T and $v_1,...v_m$ are corresponding eigen vectors. Then show that $v_1,...v_m$ is linearly independent.

(ii) Suppose $T : V \to V$ be a linear map and $\dim \textit{range } T = k$. Prove that T has at most $k+1$ distinct eigen values.

(iii) Suppose $T : V \to V$ be a linear map and there exist non zero vectors v and w in V such $Tv = 3w$ and $Tw = 3v$. Prove that 3 or –3 is an eigen value of T .

OR

(b) (i) Suppose V is a finite-dimensional complex vector and $T : V \to V$ be a linear map. Then show that T has an upper-triangular matrix with respect to some basis of V .

(ii) Suppose $T : V \to V$ be a linear map and $T^2 = I$ and 1 is not on eigen value of T . Prove that $T = -I$.

12. (a) (i) Let $T \in \mathcal{L}(V)$. Then show that the zeros of the minimal polynomial of T are precisely the eigen value of T .

(ii) Suppose $T \in \mathcal{L}(V)$ is such that $\textit{range } T^4 \neq \textit{range } T^5$. Prove that T is nilpotent.

OR

(b) (i) Let $T \in \mathcal{L}(V)$ and q be a polynomial. Then show that $q(T) = 0$ if and only if q is a polynomial multiple of the minimal polynomial of T .

(ii) Suppose N is a nilpotent operator on a finite dimensional space V . Show that there is a basis of V with respect to which the matrix of N has all entries on and below the diagonal are 0's.

13. (a) (i) Suppose $T \in L(V)$ has the same matrix with respect to every basis of V. Prove that T is a scalar multiple of the identity operator.

(ii) Suppose $T \in L(V)$ is such that $trace\,(ST) = 0$ for all $S \in L(V)$. Prove that $T = 0$.

(iii) Prove or give a counterexample. if $S, T \in L(V)$ then $trace\,(ST) = trace(S)\ trace\,(T)$.

OR

(b) (i) Suppose V is a real vector space and $T \in L(V)$ has no eigen values. Prove that $\det T > 0$.

(ii) Prove or give a counterexample. The determinant of the sum of two matrices is equal to the sum of the determinants of those matrices.

(iii) What is the determinant of a matrix with two of its columns are equal?

(iv) Suppose $S, T \in L(V)$. Then show that

$$\det(ST) = \det(TS) = \det(S)\det(T)$$

(5 × 12 = 60 Marks)
