

Reg. No. :

Name :

First Semester M.Sc. Degree Examination, August 2021.

Mathematics

MM 211 – LINEAR ALGEBRA

(2017 – 2019 Admission)

Time : 3 Hours

Max. Marks : 75

Answer either Part A or Part B of each question.

Each question carries 15 marks.

- I. (A) (i) Suppose U is a subspace of $P(F)$ consisting of all polynomials p of the form $p(z) = az^2 + bz^5$, Where $a, b \in F$. Find a subspace W of $P(F)$ such that $P(F) = U \oplus W$. 7
- (ii) If U_1 and U_2 are subspaces of a finite dimensional vector space, show that $\dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2)$. 8
- (B) (i) Prove that the real vector space consisting of all continuous functions on the interval $[0,1]$ is infinite dimensional. 5
- (ii) Suppose that U and W are both five-dimensional subspaces of R^9 . Prove that $U \cap W \neq \{0\}$. 5
- (iii) Prove that the union of two subspaces of V is a subspace of V if and only if one of the subspaces is contained in the other. 5



- II. (A) (i) Suppose that V is finite dimensional and $S, T \in L(V)$. Prove that ST is invertible if and only if both S and T are invertible. **5**
- (ii) Suppose $\{v_1, \dots, v_n\}$ is a basis of V . Prove that the function $T: V \rightarrow \text{Mat}(n, 1, F)$ defined by $Tv = M(v)$ is an invertible linear map of V onto $\text{Mat}(n, 1, F)$; here $M(v)$ is the matrix of $v \in V$ with respect to the basis (v_1, \dots, v_n) . **5**
- (iii) Suppose that $T \in L(V, W)$ is injective and $\{v_1, \dots, v_n\}$ is linearly independent in V . Prove that (Tv_1, \dots, Tv_n) is linearly independent in W . **5**
- (B) (i) Prove that if V is finite dimensional with $\dim V > 1$, then the set of noninvertible operators on V is not a subspace of $L(V)$. **5**
- (ii) Suppose that $\{v_1, \dots, v_n\}$ is a basis of V and (w_1, \dots, w_m) is a basis of W . Then show that M is an invertible linear map between $L(V, W)$ and $\text{Mat}(m, n, F)$. **5**
- (iii) Show that two finite-dimensional vector spaces are isomorphic if and only if they have the same dimension. **5**
- III. (A) (i) Show that every operator on a finite-dimensional, nonzero, real vector space has an invariant subspace of dimension 1 or 2. **8**
- (ii) Define $T \in L(F^3)$ by $T(z_1, z_2, z_3) = (2z_2, 0, 5z_3)$. Find all eigenvalues and eigenvectors of T . **7**
- (B) (i) Suppose that $S, T \in L(V)$ are such that $ST = TS$. Prove that $\text{null}(T - \lambda I)$ is invariant under S for every $\lambda \in F$. **5**
- (ii) Suppose $T \in L(V)$ and $\dim \text{range } T = k$. Prove that T has at most $k + 1$ distinct eigenvalues. **5**
- (iii) Suppose $P \in L(V)$ and $P^2 = P$. Prove that $V = \text{null } P \oplus \text{range } P$. **5**



- IV. (A) (i) If $T \in L(V)$ show that the roots of the minimal polynomial of T are precisely the eigenvalues of T . **5**
- (ii) Suppose V is a complex vector space and $T \in L(V)$. Show that there is a basis of V consisting of generalized eigenvectors of T . **5**
- (iii) If V is a complex vector space and $T \in L(V)$, then show that the sum of the multiplicities of all the eigenvalues of T equals $\dim V$. **5**
- (B) (i) Suppose V is a complex vector space and $T \in L(V)$. Prove that V has a basis consisting of eigenvectors of T if and only if the minimal polynomial of T has no repeated roots. **8**
- (ii) Suppose $T \in L(V)$ and $\{v_1, \dots, v_n\}$ is a basis of V that is a Jordan basis for T . Describe the matrix of T with respect to the basis $\{v_n, \dots, v_1\}$ obtained by reversing the order of the v 's. **7**
- V. (A) (i) Prove that if $P \in L(V)$ satisfies $P^2 = P$, then $\text{trace } P$ is a nonnegative integer. **5**
- (ii) Show that there do not exist operators $S, T \in L(V)$ such that $ST - TS = I$. **5**
- (iii) Suppose $T \in L(V)$ and (v_1, \dots, v_n) is a basis of V . Prove that $M(T, (v_1, \dots, v_n))$ is invertible if and only if T is invertible. **5**
- (B) (i) Prove or give a counter example: if $T \in L(V)$ and $c \in F$, then $\det(cT) = c^{\dim V} \det T$. **7**
- (ii) If A and B are square matrices of the same size, then show that $\det(AB) = \det(BA) = (\det A)(\det B)$. **8**

