(Pages : 4) **L – 6313**

Reg. No. : ....

Name : ....

First Semester M.Sc. Degree Examination, August 2021

Mathematics

MM 214 — TOPOLOGY – I

(2017 – 2019 Admission)

Time : 3 Hours Max. Marks : 75

Answer **all** the questions. Choosing Part A or Part B from each question.

I. (A) (a) Let (X, d) be a metric space and A a subset of X. Prove that a point x in X is a limit point of A if and only if there is a sequence of distinct points of A which converges to X. (5)

(b) For a metric space (X, d), $a \in X$ and $r > 0$, prove that the open ball $B(a,r)$ is an open set and the closed ball $B[a,r)$ is a closed set. (5)

(c) Let A be a subset of a metric space X. Prove that A is open if and only if $A = \text{int}(A)$. (5)

OR

(B) (a) Let (X, d) be a metric space and $A \subset X$. Prove that $\overline{A}$ is the smallest closed set containing A. (5)

(b) Let (X, d) be a metric space and $A \subset X$. Prove that

(i) $x \in \overline{A}$ if and only if $d(x, A) = 0$.

(ii) $x \in \text{int}(A)$ if and only if $d(x, X \setminus A) > 0$. (5)

(c) Let (X, d) be a metric space. A and B are subsets of X. if $A \subset B$, prove that $\overline{A} \subset \overline{B}$. (5)

P.T.O.

II. (A) (a) Let (X, d) be a metric space and $A \subseteq X$. Define $f: X \to \mathbb{R}$ by $f(x) = d(x.A)$, $x \in X$. Show that f is continuous. (3)

(b) State and prove Cantor's intersection theorem. (7)

(c) Prove that every contractive function is continuous. (5)

OR

(B) (a) Prove that the space $(\mathbb{R}^n, d)$ is a complete metric space, where d is the usual metric on $\mathbb{R}^n$. (5)

(b) State and prove Baire Category Theorem. (5)

(c) Prove that a complete metric space without isolated points must be un-countable. (5)

III. (A) (a) Prove that every metric space is first countable. (3)

(b) Prove that Separability is a topological property. (8)

(c) Prove that a sequence in a Hausdorff space cannot converge to more than one point. (4)

OR

(B) (a) Show that the following statements are equivalent :

(i) f is a homeomorphism.

(ii) f is an open, continuous mapping.

(iii) f is closed, continuous mapping. (6)

(b) Let (A, τ') be a subspace of a topological space (X, τ). Prove that a subset D of A is closed in the subspace topology for A if and only if $D = C \cap A$ for some closed subset C of X. (6)

(c) Prove that a finite subset of a Hausdorff X has no limit points. (3)

IV (A) Prove that the following statements are equivalent :

(a) X is connected.

(b) X is not the union of two disjoint, non-empty closed sets.

(c) X is not the union of two separated sets.

(d) there is no coutinuous function from X onto a discrete two point space $\{a, b\}$.

(e) the only subsets of X which are both open and closed are X and ϕ.

(f) X has no proper subset A for which $A \cap \overline{(X \setminus A)} = \phi$. (15)

OR

(B) (a) Let X be a space and $\{A_\alpha : \alpha \in I\}$ a family of connected subsets of X for which $\bigcap_{\alpha \in I} A_\alpha$ is non-empty. Prove that $\bigcup_{\alpha \in I} A_\alpha$ is connected. (7)

(b) Prove that connected subsets of $\mathbb{R}$ are precisely the intervals. (8)

V. (A) (a) Prove that each closed subset of a compact space is compact. (5)

(b) If A and B are disjoint compact subsets of a Hausdorff space X, then prove that there exists disjoint open sets U and V in X such that $A \subset U$ and $B \subset V$. (5)

(c) Let X be a compact space, Y a Hausdorff space, and $f : X \to Y$ a continuous function. Then prove that f is a closed function. (5)

OR

(B) (a) Let X be a compact space, Y a Hausdorff space, and $f : X \to Y$ a continuous function from X onto Y. Then prove that f is a homeomorphism. (6)

(b) Let $X = [0,1)$ and let $Y = S'$, the unit circle in $\mathbb{R}^2$. Prove that the function defined by

$$f(x) = (\cos 2\pi x, \sin 2\pi x), \ x \in [0,1)$$

is (i) continuous, (ii) one-to-one (iii) f^{-1} is not continuous. (3)

(c) State and prove Lindel of Theorem. (6)
