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**Reg. No. : ..............................**

**Name : ..................................**

**First Semester M.Sc. Degree Examination, August 2021.**

**Mathematics**

**MM 214 – TOPOLOGY I**

**(2020 Admission)**

Time : 3 Hours Max. Marks : 75

PART – A

Answer **any five** questions from among the questions 1 to 8. Each question carries **3** marks.

1. For a metric space $(X, d), a \in X$, and $r > 0$ prove that the open ball $B(a, r)$ is all open set.

2. Give an example of two subsets A and B of a metric space. $X$ for which $A \subset B$ but neither *bdy A nor bdy B* is a subset of the other.

3. Is every Cauchy sequence convergent? Justify your answer.

4. Give an example of a set X with two equivalent metrics d and d' for which (X, d) is complete and $(X, d')$ is not.

5. For $X = \{1,2\}$ with discrete topology and $A = \{1\}$, find *int A, bdy A and* $A'$.

6. Give an example of a function that is both open and closed but not, continuous.

7. Prove that a discrete space with more than one point is disconnected.

8. Give an example of a compact subspace of a topological space that is not closed.

**(5 × 3 = 15 Marks)**

**P.T.O.**

PART – B

Answer **all** questions from 9 to 13. Each question carries **12** marks.

9. (a) (i) Define the taxicab metric $d'$ on $\mathbb{R}^n$ . Show that it is a metric on $\mathbb{R}^n$

(ii) Show that a sequence in a metric space cannot converge to more than one limit.

(iii) Let A be a subset of a metric space $X$ . Prove that, A is closed if and only if $A = \overline{A}$.

OR

(b) (i) State and prove Cauchy-Schwarz inequality.

(ii) Prove that a subset of a metric space $(X, d)$ is closed if and only if A contains all of its limit points.

(iii) For a subset A of a metric space $X$ , prove that $x \in \overline{A}$ if and only if $d(x, A) = 0$ .

10. (a) (i) Let $f : X \to Y$ be a function from metric space $(X, d)$ to metric space $(Y, d')$ Prove that f is continuous if and only if for each sequence $\{x_n\}_{n=1}^{\infty}$ converging to a point a in X, the sequence $\{f(x_n)\}_{n=1}^{\infty}$ converges to $f(a)$ .

(ii) State and prove Baire category theorem.

(iii) Define isometry between two metric spaces. Give an example of different isometric metric spaces.

OR

(b) (i) Let $f : X \to Y$ be a function on the indicated metric spaces and let a be a point, of X. Prove that f is continuous at a if and only if for each open set O containing $f(a), f^{-1}(O)$ is a neighbourhood of a.

(ii) Show that every function $f : X \to Y$ for which the domain X has the discrete metric is continuous.

(iii) Let $(X, d)$ be a complete metric space. Prove that a subspace A of X is complete if and only if it is closed.

11. (a) (i) For any subsets A, B of a topological space X prove that $\text{int}(A \cap B) = \text{int}\, A \cap \text{int}\, B$.

(ii) Prove that separability is a topological property.

(iii) Prove that the property being a Hausdorff space is a topological and hereditary property.

OR

(b) (i) Show that the finite complement topology is actually a topology for any set X.

(ii) Prove that every separable metric space is second countable.

(iii) If $f : X \to Y$ and $g : Y \to Z$ are continuous functions on the indicated spaces, then prove that the composite function $g \circ f : X \to Z$ is continuous.

12. (a) (i) Define a disconnect topological space. Prove that a topological space X is disconnected if and only if X is the union of two disjoint, non-empty closed sets.

(ii) Prove that the fixed-point property is topological invariant property. Show that the real line does not have the fixed-point property.

(iii) Prove that $\mathbb{R}^n$ is locally path connected.

OR

(b) (i) Let $X$ be a space and $\{A_\alpha : \alpha \in I\}$ a family of connected subsets of X for which $\cap_{\alpha \in I} A_\alpha$ is not empty. then prove that $\{\cup_{\alpha \in I} A_\alpha\}$ is connected.

(ii) State and prove the intermediate value theorem.

(iii) Prove that every open, connected subset of $\mathbb{R}^n$ is path connected.

13. (a) (i) Prove that a space X is compact if and only if every family of closed sets in X with finite intersection property has non-empty intersection.

(ii) Let $X$ be a compact space, Y a space and $f : X \to Y$ a continuous function from $X$ onto $Y$. Then show that Y is compact.

OR

(b) (i) Let $X$ be a compact hausdorff space. prove that a subset A of X is compact if and only if it is closed.

(ii) Let $X$ be a compact space and $f : X \to \mathbb{R}$ a continuous real-valued function on $X$. The prove that there are members $c$ and $d$ of $X$ such that, for all $x \in X$ $f(c) \leq f(x) \leq f(d)$.

(iii) Prove that every compact space has the Bolzano-Weierstrass property.

**(5 $\times$ 12 = 60 Marks)**

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