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Reg. No. : ....

Name : ....

First Semester M.Sc. Degree Examination, August 2021.

Mathematics

MM 211 – LINEAR ALGEBRA

(2017 – 2019 Admission)

Time : 3 Hours Max. Marks : 75

Answer either Part A or Part B of each question.

Each question carries 15 marks.

I. (A) (i) Suppose U is a subspace of $P(F)$ consisting of all polynomials p of the form $p(z) = az^2 + bz^5$, Where $a, b \in F$. Find a subspace W of $P(F)$ such that $P(F)$ such that $P(F) = U \oplus W$. **7**

(ii) If U_1 and U_2 are subspaces of a finite dimensional vector space, show that $\dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2)$. **8**

(B) (i) Prove that the real vector space consisting of all continuous functions on the interval [0,1] is infinite dimensional. **5**

(ii) Suppose that U and W are both five-dimensional subspaces of R^9. Prove that $U \cap W \neq \{0\}$. **5**

(iii) Prove that the union of two subspaces of V is a subspace of V if and only if one of the subspaces is contained in the other. **5**

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II. (A) (i) Suppose that V is finite dimensional and $S,T \in L(V)$. Prove that ST is invertible if and only if both S and T are invertible. **5**

(ii) Suppose $\{v_1,....,v_n\}$ is a basis of V. Prove that the function T: $V \to Mat(n,1,F)$ defined by $Tv = M(v)$ is an invertible linear map of V onto $Mat(n,1,F)$; here $M(v)$ is the matrix of $v \in V$ with respect to the basis $(v_1,..,v_n)$. **5**

(iii) Suppose that $T \in L(V,W)$ is injective and $\{v_1,....,v_n\}$ is linearly independent in V. Prove that $(Tv_1,..,Tv_n)$ is linearly independent in W. **5**

(B) (i) Prove that if V is finite dimensional with dim V > 1, then the set of noninvertible operators on V is not a subspace of L(V). **5**

(ii) Suppose that $\{v_1,....,v_n\}$ is a basis of V and $(w_1,....w_m)$ is a basis of W. Then show that M is an invertible linear map between L(V,W) and $Mat(m,n,F)$. **5**

(iii) Show that two finite–dimensional vector spaces are isomorphic if and only if they have the same dimension. **5**

III. (A) (i) Show that every operator on a finite–dimensional, nonzero, real vector space has an invariant subspace of dimension 1 or 2. **8**

(ii) Define $T \in L(F^3)$ by $T(z_1, z_2 z_3) = (2z_2, 0, 5z_3)$. Find all eigenvalues and eigenvectors of T. **7**

(B) (i) Suppose that $S,T \in L(V)$ are such that $ST = TS$. Prove that null $(T - \lambda I)$ is invariant under S for every $\lambda \in F$. **5**

(ii) Suppose $T \in L(V)$ and dim range $T = k$. Prove that T has at most $k+1$ distinct eigenvalues. **5**

(iii) Suppose $P \in L(V)$ and $P^2 = P$. Prove that $V = null\, P \oplus range\, P$. **5**

IV. (A) (i) If $T \in L(V)$ show that the roots of the minimal polynomial of T are precisely the eigenvalues of T. **5**

(ii) Suppose V is a complex vector space and $T \in L(V)$. Show that there is a basis of V consisting of generalized eigenvectors of T. **5**

(iii) If V is a complex vector space and $T \in L(V)$, then show that the sum of the multiplicities of all the eigenvalues of T equals dim V. **5**

(B) (i) Suppose V is a complex vector space and $T \in L(V)$. Prove that V has a basis consisting of eigenvectors of T if and only if the minimal polynomial of T has no repeated roots. **8**

(ii) Suppose $T \in L(V)$ and $\{v_1,....,v_n\}$ is a basis of V that is a Jordan basis for T. Describe the matrix of T with respect to the basis $\{v_n,....,v_1\}$ obtained by reversing the order of the $v's$. **7**

V. (A) (i) Prove that if $P \in L(V)$ satisfies $P^2 = P$, then trace P is a nonnegative integer. **5**

(ii) Show that there do not exist operators $S, T \in L(V)$ such that $ST - TS = I$. **5**

(iii) Suppose $T \in L(V)$ and $(v_1,....,v_n)$ is a basis of V. Prove that $M(T,(v_1,...v_n))$ is invertible if and only if T is invertible. **5**

(B) (i) Prove or give a counter example: if $T \in L(V)$ and $c \in F$, then $\det(cT) = c^{\dim V} \det T$. **7**

(ii) If A and B are square matrices of the same size, then show that $\det(AB) = \det(BA) = (\det A)(\det B)$. **8**