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Reg. No. : .............................

Name : .................................

## Fourth Semester M.Sc. Degree Examination, May 2020

## Branch : Mathematics

## MM 242 : FUNCTIONAL ANALYSIS — II

## (2017 Admission Onwards)

Time : 3 Hours Max. Marks : 75

Answer either Part A or Part B of each question.

All questions carry equal marks.



I. (A) (a) Let X be a normed space and $A \in CL(X)$. Prove that $\sigma_a(A) = \sigma(A)$. **7**

(b) Let X be a normed space and $A \in CL(X)$. Show that the eigen-spectrum and the spectrum of A are countable sets and have 0 as the only possible limit point. **8**

(B) (a) Let X be a normed space and $A \in CL(X)$. Prove that

$$\{k : k \in \sigma_e(A), k \neq 0\} = \{k : k \in \sigma(A), k \neq 0\}.$$ **7**

(b) Show that 0 can be a spectral value of a compact operator A without being its eigenvalue and can be the limit point of the spectrum of A. **8**

II. (A) (a) State and prove the Riesz-Fischer theorem. **8**

(b) Let $\langle , \rangle$ be an inner product on a linear space X and $T : X \to X$ be a linear map. Let $\langle x, y \rangle_T = \langle T(x), T(y) \rangle$, $x, y \in X$. Show that $\langle , \rangle_T$ is an inner product on X if and only if T is one-to-one. **7**

P.T.O.

(B) (a) State and prove Gram-Schmidt orthonormalization theorem. **8**

(b) Let X be an inner product space. Show that the closed unit ball in X is compact if and only if X is finite dimensional. **7**

III. (A) (a) State the projection theorem and the Riesz representation theorem and show that they do not hold for an incomplete inner product space. **7**

(b) State and prove the unique Hahn-Banach extension theorem. **8**

(B) Show that a subset of a Hilbert space is weak bounded if and only if it is bounded. **15**

IV. (A) (a) State and prove the generalized Schwarz inequality. **8**

(b) Let $A \in BL(H)$ be self-adjoint. Show that $A^2 \geq 0$ and $A \leq \|A\| I$. If $A^2 \leq A$, Show that $0 \leq A \leq I$. **7**



(B) Let H be a Hilbert space and $A \in BL(H)$. Then prove the following:

(a) $Z(A) = R(A^*)^{\perp}$ and $Z(A^*) = R(A)^{\perp}$. Moreover, A is injective if and only if $R(A^*)$ is dense in H and A* is injective if and only if $R(A)$ is dense in H. **5**

(b) The closure of $R(A)$ equals $Z(A^*)^{\perp}$ and the closure of $R(A^*)$ equals $Z(A)^{\perp}$. **5**

(c) $R(A) = H$ if and only if $A^*$ is bounded below and $R(A^*) = H$ if and only if A is bounded below. **5**

V. (A) Let H be a Hilbert space and $A \in BL(H)$. Then prove the following:

(a) If $R(A)$ is finite dimensional, then A is compact. **5**

(b) If each $A_n$ is a compact operator on H and $\|A_n - A\| \to 0$, then A is compact. **5**

(c) If A is compact, then so is $A^*$. **5**

(B) Let $A$ be a compact operator on $H \neq \{0\}$

(a) Show that every nonzero approximate eigenvalue of $A$ is, in fact, an eigenvalue of $A$ and the corresponding eigenspace is finite dimensional. **8**

(b) If A is self-adjoint, then show that $\|A\|$ or $-\|A\|$ is an eigen value of A. **7**