(Pages : 3) **L – 5424**

Reg. No. : ....

Name : ....

Fourth Semester M.Sc. Degree Examination, March 2021

Mathematics

MM 244 Elective : ANALYTIC NUMBER THEORY

(2005 Admission Onwards)

Time : 3 Hours Max. Marks : 75

Answer either Part A or Part B of each question.

Each question carries **15** marks.

1. (A) (a) Prove that if $(a,b) = d$ and $(b,c) = f$, then $(d,c) = (a,f)$. **3**

(b) Prove that every $n > 1$ can be represented as a product of primes in only one way, apart from the order of the factors. **8**

(c) Prove that if $n > 1$, then $n^4 + 4$ is composite. **4**

OR

(B) (a) Prove that there are infinitely many prime numbers. **4**

(b) Prove that series $\sum_{n=1}^{\infty} \frac{1}{p_n}$ diverges, when P_n denotes the n[th] prime (positive). **7**

(c) Prove that if $(a,b) = 1$, then there exists $x > 0$ and $y > 0$ such that $ax - by = 1$. **4**

P.T.O.

2. (A) (a) Prove that $\sum_{d/n} \varphi(d) = n$, for $n \geq 1$. **8**

(b) Give an example of a function which is multiplicative but not completely multiplicative. Justify your claim. **3**

(c) Let f be a multiplicative arithmetical function with $f(1) = 1$. Prove that f is completely multiplicative if and only if $f\left(p^a\right) = (f(p))^a$, for all primes p and all integers $a \geq 1$. **4**

OR

(B) (a) State and prove the generalized inversion formula. **5**

(b) Define the Liouville's function $\lambda(n)$. Also find the bell series of λ modulo p. **5**

(c) Prove that if f is multiplicative, then $f^{-1}(n) = \mu(n)\, f(n)$ for every square free n. **5**

3. (A) (a) Prove that for any positive integer m, the m residue classes $\hat{1}, \hat{2}, \ldots \hat{m}$ are disjoint and their union is the set of all integers. **6**

(b) Verify whether the following linear congruences have a solution or not. If yes, find all its solution.

(i) $12x \equiv 5 (\text{mod } 18)$

(ii) $12x \equiv 4 (\text{mod } 18)$. **5**

(c) Prove that if $(a, m) = 1$, then $a^{\phi(m)} \equiv 1 (\text{mod } m)$. **4**

OR

(B) (a) Let $f(x) = c_0 + c_1x + \ldots c_nx^n$ be a polynomial of degree n with integer coefficients such that $c_n \not\equiv 0 (\text{mod } p)$. Prove that $f(x) \equiv 0 (\text{mod } p)$ has at most n solutions. **8**

(b) Let r, d and k be integers such that d / k, $d > 0$, $k \geq 1$ and $(r, d) = 1$. Prove that the numbers of elements in the set $\{r + td : t = 1, 2, \ldots, k / d\}$. Which are relatively prime to k is $\frac{\phi(k)}{\phi(d)}$ **7**

4. (A) (a) Evaluate

(i) $(11/13)$ and

(ii) $(12/13)$. **3**

(b) Let m be the least positive residues mod p of the numbers $n, 2n, 3n, \ldots, \left(\frac{p-1}{2}\right)n$, which exceed $p/2$. Prove that

$$m = \sum_{t=1}^{(p-1)/2} \left[\frac{tn}{p}\right] + (n-1)\left(\frac{p^2-1}{8}\right) (\mathrm{mod}\, 2).$$ **12**

OR

(B) (a) Prove that if P and Q are positive odd integers with $(P, Q) = 1$, then

$$(P/Q)(Q/P) = (-1)^{(P-1)(Q-1)/4}.$$ **6**

(b) Prove that the diophantine equation $y^2 = x^3 + k$ has no solution if k has the form $k = (4n-1)^3 - 4m^2$, where m and n are integers such that no prime $p \equiv -1 (\mathrm{mod}\, 4)$ division m. **9**

5. (A) (a) Prove that there are no primitive roots mod 2^{α} for $\alpha \geq 3$. **5**

(b) If p is an odd prime and g is a primitive root mod p such that $g^{p-1} \not\equiv 1 (\mathrm{mod}\, p^2)$, then prove that for every $\alpha \geq 2$, $g^{\phi(p^{\alpha-1})} \not\equiv 1 (\mathrm{mod}\, p^{\alpha})$. **7**

(c) Prove that m is a prime if and only if $\exp_m(a) = m-1$ for same a. **3**

OR

(B) Prove that there are no primitive roots mod m, if m is a positive integer not of the form $m = 1, 2, 4, p^{\alpha}$ or $2p^{\alpha}$ where p is an odd prime. **15**

(5 × 15 = 75 Marks)
