(Pages : 3) **L – 5419**

Reg. No. : ....

Name : ....

Fourth Semester M.Sc. Degree Examination, March 2021

Mathematics

MM 243 Elective : FIELD THEORY

(2013 Admission onwards)

Time : 3 Hours Max. Marks : 75

All questions carry equal marks.

1. (A) (a) Prove that A_5 is a simple group. **5**

(b) Prove that the commutator subgroup of S_n of A_n. **5**

(c) If G is solvable group, prove that every subgroup and every quotient group of G is also solvable. **5**

OR

(B) (a) Prove that every finite abelian group $G \neq \{1\}$ contains a sub-group of prime index.

(b) If α is a 5-cycle in S_5 and τ is a transposition in S_5, prove that $\langle \sigma, \tau \rangle = S_5$.

2. (A) (a) Find all prime subfields of a field. **8**

(b) If F is a field, prove that a nonzero polynomial $p(x) \in F[x]$ is irreducible if and only if $\langle p(x) \rangle$ is a prime ideal. **7**

OR

P.T.O.

(B) (a) State and prove Eisenstein Criterion. **8**

(b) Prove that the p^{th} Cyclotomic polynomial $\phi_p(x)$ is irreducible in $\mathbb{Q}[x]$ for every prime p. **7**

3. (A) (a) Prove that a finite field extension of a finite field extension is finite field extension. **7**

(b) Let $p(x) \in F[x]$ be an irreducible polynomial of degree d. Prove that $E = F[x]/\langle p(x) \rangle$ is a field extension of F of degree d and E contains a root α of $p(x)$. **8**

OR

(B) (a) If $(x) \in F[x]$, prove that any two splitting fields of $f(x)$ over F are isomorphic by an isomorphism fixing F point wise. **9**

(b) Find the Galois group of $x^4 + 1$ over $\mathbb{Q}$. **6**

4. (A) (a) If F is field with multiplicative group $F^* = F - \{0\}$, prove that every finite subgroup G of F^* is cyclic. **7**

(b) Prove that $Gal(GF(p^n) \mid GF(p)) \cong \square_n$ for all prime p. **8**

OR

(B) (a) If p is a prime, F is a field of characteristic zero, $f(x) = x^p - c$, prove that either $f(x)$ is irreducible in $F[x]$ or c has a p^{th} root in F. **8**

(b) Prove that there exists a quintic polynomial $f(x) \in \mathbb{Q}[x]$ that is not solvable by radicals. **7**

5. (A) (a) Let $E \mid F$ be a Galois Extension, and let B be an intermediate field. Prove that the following conditions are equivalent :

(i) B has no conjugates

(ii) if $\sigma \in Gal(E \mid F)$, then $\sigma_{|B} \in Gal(B \mid F)$.

(iii) $B \mid F$ is a Galois extension. **10**

(b) If $E \mid F$ is a Galois extension and B is an intermediate field, then $E \mid B$ is a Galois extension. **5**

OR

(B) (a) State and prove Hilbert's theorem. **9**

(b) Prove that the Galois field $GF(p^n)$ has exactly one subfield of order p^d for every divisor d of n. **6**

(5 × 15 = 75 Marks)