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Reg. No. : ....

Name : ....

Fourth Semester M.Sc. Degree Examination, September 2019

Mathematics

MM 242 – FUNCTIONAL ANALYSIS – II

(2017 Admission)

Time : 3 Hours Max. Marks : 75

Answer either Part A or Part B of each question.

All questions carry equal marks.

I. (A) Let $A \in CL(X)$, where X is a normed space. Then prove that

(a) The eigen spectrum and the spectrum of A are countable sets and have 0 as the only possible limit point. 8

(b) Eigen spaces corresponding to non zero eigen values of A are finite dimensional. 7

OR

(B) Let X be a normed space, $A \in BL(X)$ and $A(x_n) = k_n x_n$ for some $x_n \neq 0$ in X and $K_n \in, \mathbb{K}, n = 1, 2, 3,..$

(a) Suppose $k_n \neq k_m$ whenever $n \neq m$. Then prove that $\{x_1, x_2, x_3, ...\}$ is linearly independent. 7

(b) Suppose $k_n \neq k_m$ whenever $n \neq m$ and A is compact. If the set $\{x_1, x_2, x_3 ...\}$ is infinite prove that $k_n \to 0$ as $n \to \infty$. 8

P.T.O.

II. (A) Let X be an inner product space. Then, prove the following :

(a) Polarization identity. 5

(b) For $x \in X$, $x = 0$ if and only if $< x, y >= 0$ for all $y \in x$. 2

(c) Schwarz inequality. 8

OR

(B) Let X be an inner product space over $\mathbb{K}$, $\{u_1, u_2, ...\}$ be a countable orthonormal set in X and $k_1, k_2, ... \in \mathbb{K}$.

(a) If $\sum\limits_{n} k_n u_n$ converges to some $x \in X$, then $< x, u_n >= k_n$ for all n and $\sum\limits_{n=1}^{\infty} |k_n|^2 = \|x\|^2$. 7

(b) State and prove the Riesz-Fischer theorem. 8

III. (A) Let X be an inner product space $\{x_1, x_2, ...x_m\}$ be a linearly independent subset of X and $x \in X$.

(a) Let $F = span\{x_1, x_2, ..., x_m\}$. Find the unique best approximation from F to x. 8

(b) Let $c_1, c_2, ..., c_m$ be scalars and $E = \{y \in X / < y, x_j > = c_j$ for $j = 1, 2, ..., m\}$. Find the unique best approximation from E to x. 7

OR

(B) (a) State and projection theorem and the Riesz representation theorem. 10

(b) Give examples to show that projection and Riesz representation theorems need not be true in incomplete inner product spaces. 5

IV. (A) Let H be a Hilbert space and $A \in BL(H)$. Then prove the following :

(a) $z(A) = R(A^*)^\perp$ and

$z(A^*) = R(A)^\perp$. 5

(b) $\overline{R(A)} = z(A^*)^\perp$ and

$\overline{R(A^*)} = z(A)^\perp$. 5

(c) $R(A) = H$ if and only if A^* is bounded below and

$R(A^*) = H$ if and only if A is bounded below 5

OR

(B) (a) Define normal, unitary and self-adjoint operators. Show by an example that a normal operator need not be neither unitary nor self-adjoint. 5

(b) Let E be a measurable subset of $\mathbb{R}$. Let $H = L^2(E)$. Fix $z \in L^\infty(E)$ and define $A(x) = zx$ for $x \in H$. Show that A is a normal operator on H. Find conditions under which A is unitary and self-adjoint. 5

(c) Let $\{u_1, u_2 ....\}$ be a countable orthonormal basis for a separable Hilbert space H and (k_n) be a bounded sequence of scalars.

Let $A(x) = \sum_n K_n < x, u_n > u_n$, for $x \in H$.

Show that A is a normal operator on H. Find conditions under which A is unitary and self-adjoint. 5

V. (A) Let A be a self-adjoint operator on a non zero Hilbert space H.

(a) Prove that $\{m_A, M_A\} \subset \sigma_a(A) = \sigma(A) \subset [m_A, M_A]$. 8

(b) Establish the Ritz method. 7

OR

(B) (a) Define a compact operator and give an example. 3

(b) Let A be a compact operator on a non zero Hilbert space H. Show that every non zero approximate eigenvalue of A is an eigenvalue and the corresponding eigenvalue is finite-dimensional. 6

(c) Let A be a self-adjoint compact operator on a non zero Hilbert space H. Show that $\|A\|$ or $-\|A\|$ is an eigenvalue of A. 6