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Reg. No. : ....

Name : ....

Fourth Semester M.Sc. Degree Examination, September 2019

Mathematics

MM 242 : FUNCTIONAL ANALYSIS – II

(Prior to 2017 Admission)

Time : 3 Hours Max. Marks : 75

All questions carry equal marks.

1. (A) (i) Show that for all $x, y \in X$, $|\langle x, y\rangle|^2 \leq \langle x, x\rangle \langle y, y\rangle$, where X is an inner product space.

(ii) State and prove Gram-Schmidt orthonormalization theorem.

(iii) State and prove Bessel's inequality. **5+5+5**

OR

(B) If H is a non zero Hilbert space over K, show that the following conditions are equivalent :

(i) H has a countable orthonormal basis;

(ii) H is linearly isometric to K^n for some n, or to l^2;

(iii) H is separable. **15**

P.T.O.

2. (A) (i) If F is a subspace of an inner product space and $x \in X$, show that $y \in F$ is a best approximation from F to x if and only if $x - y \perp F$ and in that case

$$dist(x, F) = \langle x, x - y \rangle^{\frac{1}{2}}.$$

(ii) Prove the existence and the uniqueness of a best approximation from a non empty closed convex subset E of a Hilbert space H to a point $x \in H$. **7+8**

OR

(B) (i) If H is a Hilbert space and F is a non empty closed subspace of H, prove that $H = F + F^{\perp}$.

(ii) State and prove Riesz representation theorem. **5+10**

3. (A) (i) If H is a Hilbert space and $A \in BL(H)$, show that there is a unique $B \in BL(H)$ such that for all $x, y \in H$, $\langle A(x), y \rangle = \langle x, B(y) \rangle$.

(ii) If H is a Hilbert space and $A \in BL(H)$, show that $R(A) = H$ if and only if A^* is bounded below, and $R(A^*) = H$ if and only if A is bounded below. **5+10**

OR

(B) (i) If H is a Hilbert space and $A \in BL(H)$, show that A is normal if and only if $\|A(x)\| = \|A^*(x)\|$ for all $x \in H$. In that case, also show that $\|A^2\| = \|A^*A\| = \|A\|^2$.

(ii) Let H be a Hilbert space, (A_n) be a sequence in $BL(H)$ and $A \in BL(H)$ be such that $\|A_n - A\| \to 0$ as $n \to \infty$. If each A_n is self-adjoint, unitary or normal, then prove that A is self-adjoint, unitary or normal, respectively. **5+10**

4. (A) (i) If H is a Hilbert space and $A \in BL(H)$, prove that $\sigma_e(A) \subset \sigma_a(A)$ and $\sigma(A) = \sigma_a(A) \cup \{\bar{k} \in K \mid k \in \sigma(A)\}^a$.

(ii) If $A \in BL(H)$, show that $\sigma_e(A) \subset w(A)$, the numerical range of A and $\sigma(A)$ is contained in the closure of $w(A)$. **9+6**

OR

(B) (i) State and prove the finite dimensional spectral theorem for self-adjoint or normal operators.

(ii) If $A \in BL(H)$ is a Hilbert-Schmidt operator, prove that A is compact. **10+5**

5. (A) (i) Show that the mapping $x \mapsto x^{-1}$ of G into G is continuous and is therefore a homeomorphism of G onto itself.

(ii) If x is an element of a Banach algebra A, prove that the spectrum $\sigma(x)$ of x is a non-empty compact set in $\mathbb{C}$. **7+8**

OR

(B) (i) If x is an element of a Banach algebra A, prove that the spectral radius of x is $r(x) = \lim \|x^n\|^{\frac{1}{n}}$.

(ii) If $1 - xr$ is regular, then prove that $1 - rx$ is also regular. **10+5**
