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E – 3847

Reg. No. : ....

Name : ....

Fourth Semester M.Sc. Degree Examination, July 2018
Branch : MATHEMATICS
MM 242 : Functional Analysis – II
(2009 Admission Onwards)

Time : 3 Hours Max. Marks : 75

Instruction : *Answer either Part* ***A*** *or Part* ***B*** *of each question.* ***All*** *questions carry* ***equal*** *marks.*

UNIT – I

I. A) a) Let X be a linear space with an inner product $<\ ,\ >$. Prove that for all $x, y \in X$, $|<x, y>|^2 \leq <x, x><y, y>$, where equality holds if and only if the set $\{x, y\}$ is linearly dependent. 5

b) Let X be a linear space with an inner product $<,>$. For $x \in X$, define $\|x\| = <x, x>^{1/2}$. Prove that $\|\ \| : X \to \mathbb{K}$ is a norm on X. 4

c) Let $X = C_{00}$, the linear space of all scalar sequences each of which has only a finite number of nonzero entries. For $x, y \in X$, define $<x, y> = \sum_{j=1}^{\infty} x(j)\,\overline{y(j)}$. Prove that X is an inner product space which is not a Hilbert space and the completion of X is l^2. 6

B) a) State and prove Bessel's inequality. 4

b) Using Gram-Schmidt orthogonalisation process, find an orthonormal basis for $L^2[-1,1]$. 5

c) Let H be a nonzero Hilbert space over $\mathbb{K}$. Prove that the following conditions are equivalent :

i) H has a countable orthonormal basis.

ii) H is linearly isometric to $\mathbb{K}^n$ for some n, or to l^2.

iii) H is separable. 6

P.T.O.

UNIT – II

II. A) a) Let X be an inner product space and Let F be a subspace of X with $x \in X$. Prove that $y \in F$ is a best approximation from F to x if and only if $x - y \perp F$ and in that case dist $(x, F) = < x, x - y >^{1/2}$. 8

b) Minimize $\int_0^1 |y|^2 \, dm$, subject to $y \in L^2([0, 1])$, $\int_0^1 ty(t) \, dm(t) = 1$ and $\int_0^1 t^2 y(t) \, dm(t) = 2$. 7

B) a) State and prove the unique Hahn-Banach extension theorem. 8

b) Prove that a subset of a Hilbert space is weakly bounded if and only if it is bounded. 7

UNIT – III

III. A) a) Let H be a Hilbert space and $A \in BL(H)$. Prove that there is a unique $B \in BL(H)$ such that for all $x, y \in H$, $< A(x), y > = < x, B(y) >$. 7

b) Let H be a Hilbert space and $A \in BL(H)$. Prove the following :

i) $Z(A) = R(A^*)^{\perp}$

ii) The closure of R(A) equals $Z(A^*)^{\perp}$

iii) R(A) = H if and only if A^* is bounded below. 8

B) a) Let H be a Hilbert space $A \in BL(H)$. Prove the following :

i) If A is self-adjoint, then A = 0 if and only if $< A(x), x > = 0$ for all $x \in H$.

ii) A is unitary if and only if $\|A(x)\| = \|x\|$ for all $x \in H$.

iii) A is normal if and only if $\|A(x)\| = \|A^*(x)\|$ for all $x \in H$. 8

b) State and prove the generalized Schwarz inequality. 7

UNIT – IV

IV. A) a) Let H be a Hilbert space and let $A \in BL(H)$. Then prove the following :

i) $k \in \sigma(A)$ if and only if $\bar{k} \in \sigma(A^*)$.

ii) If $\sigma_e(A)$ is the eigen spectrum of A, $\sigma_\alpha(A)$ is the approximate eigen spectrum of A and $\sigma(A)$ is the spectrum of A, prove that $\sigma_e(A) \subseteq \sigma_\alpha(A)$ and $\sigma(A) = \sigma_\alpha(A) \cup \{k : \bar{k} \in \sigma_\alpha(A^*)\}$. 8

b) State and prove the finite dimensional spectral theorem for self adjoint or normal operators. 7

B) a) Let $A \in BL(H)$ be a Hilbert-Schmidt operator. Then prove the following :

i) A is compact

ii) A^* is a Hilbert-Schmidt operator. 7

b) Let A be a nonzero compact self-adjoint operator on a Hilbert space H over K. Prove that there exist a finite or infinite sequence $\{s_n\}$ of nonzero real numbers with $|s_1| \geq |s_2| \geq \ldots$ and an orthonormal set $\{u_1, u_2, \ldots\}$ in H such that $A(x) = \Sigma_n s_n (x, u_n) u_n$, $x \in H$. Also prove that if the set $\{u_1, u_2, \ldots\}$ is infinite, then $s_n \to 0$ as $n \to \infty$. 8

UNIT – V

V. A) a) Let A be a Banach Algebra and let G, S be the set of all regular and singular elements of A respectively. Prove the following :

i) The mapping $x \to x^{-1}$ of G onto G is a homeomorphism.

ii) If Z is the set of all topological divisors of zero, then the boundary of S is a subset of Z. 7

b) Derive the spectral radius formula. 8

B) a) Let A be a Banach Algebra and let $x \in A$. Define $\sigma(x)$ and show that it is non-empty. 7

b) Let A be a Banach Algebra. Define the radical of A and prove that it is a proper closed two sided ideal. 8