(Pages : 7) **M – 1564**

Reg. No. : ....

Name : ....

Fifth Semester B.Sc. Degree Examination, December 2021

First Degree Programme under CBCSS

Statistics

Core Course V

ST 1541 – LIMIT THEOREMS AND SAMPLING DISTRIBUTIONS

(2018 and 2019 Admission)

Time : 3 Hours Max. Marks : 80

Use of scientific calculators and statistical tables are allowed.

SECTION – A

Answer **all** questions. Each question carries **1** mark.

1. Using axioms of probability show that $P(A^c) = 1 - P(A)$.

2. Suppose $\{X_n\}$ and $\{Y_n\}$ converges to X and Y respectively in probability. What can you say about the convergence of $\{X_n \cdot Y_n\}$?

3. Define convergence in distribution of a sequence of random variables $\{X_n\}$.

4. What is the sample range of a random sample $X_1, X_2, \ldots, X_n$ drawn from a population?

P.T.O.

5. If $\overline{X}$ is the sample mean of a random sample drawn from a population with mean μ and variance σ^2, what is the Mean Square Error of $\overline{X}$?

6. If $X^2 \sim X_n^2$, where $n > 2$, obtain the point at which the probability density function of χ^2 attains maximum?

7. If $\chi_1^2 \sim \chi_{(3)}^2$ and $\chi_2^2 \sim \chi_{(5)}^2$ are two independent Chi-square random variables, what is the mean of $\chi_1^2 + \chi_2^2$?

8. If $t \sim t_{(4)}$ is a student's t variable with 4 degrees of freedom, using statistical table, find k such that $P(|t| \leq k) = 0.90$.

9. Let (X_1, X_2, X_3) be a random sample from $N(\mu, \sigma^2)$ then define a F statistic with (1, 2) degrees freedom using (X_1, X_2, X_3).

10. Define non-central F distribution.

(10 × 1 = 10 Marks)

SECTION – B

(Answer **any eight** Questions. Each question carries **2** marks)

11. Let $C_1, C_2, \ldots\ldots C_n, \ldots\ldots$ be a partition of a sample space and if A is any event, then prove that $P(A) = \sum_{i=1}^{\infty} P(A \cap C_i)$

12. Suppose $\{X_n\}$ is a sequence of random variables with probability mass function $P(X_n = 1) = \frac{1}{n}$ and $P(X_n = 0) = 1 - \frac{1}{n}$. Examine the convergence in probability of $\{X_n\}$.

13. Let $\{X_n\}$ be a sequence of random variables with distribution function $F_{X_n}(x) = 1 - \left(1 - \frac{x}{n}\right)^n$; $x > 0$ and 0 otherwise. Show that $\{X_n\}$ converges in distribution to an exponential distribution with unit mean.

14. If $\{A_n\}$, $n = 1, 2, ...$ is a sequence of events defined over a probability space, define independence of events.

15. A random sample of size 64 are drawn from a population with mean 32 and standard deviation 5. Find the mean and standard deviation of the sample mean $\overline{X}$.

16. Let $X_1, X_2, ... X_n, X_{n+1}$ be a random sample of size $n+1$ then show that $\overline{X}_{n+1} = \frac{X_{n+1} + n\overline{X}_n}{n+1}$, where $\overline{X}_{n+1}$ and $\overline{X}_n$ are the sample means of first $n+1$ and n observations respectively.

17. A random variable X has mean 5 and variance 3. Find the least value of $P(|X - 5| < 7.5)$ using Chebyshev's inequality.

18. Let $X_1, X_2, ..., X_{100}$ be a random sample of size 100 drawn from a population with mean 10 and variance 9, then find $P(\overline{X} > 10.5)$ using central limit theorem.

19. Let $(X_1, X_2, ... X_n)$ be a random sample drawn from a population with distribution function $F(x.)$ Find the distribution function of smallest order statistic $X_{(1)}$.

20. Let $X_1, X_2, ..., X_{20}$ is a random sample of size 20 drawn from a normal population with mean μ and variance $\sigma^2 = 5$, if $s^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2$ is the sample variance, find the mean of s^2.

21. If (X_1, X_2, X_3) is a random sample of size 3 from a standard normal population $N(0,1)$, what is the sampling distribution of $U = \frac{\sqrt{2}X_3}{\sqrt{X_1^2 + X_2^2}}$.

22. Suppose $X \sim \chi^2_{(n)}$ and $Z = X + Y \sim \chi^2_{(m)}$ where X and Y are independent random variables and $m > n$. Write down the moment generating function of X and Z. Hence identify the distribution of Y.

23. Let $X_i \sim N(i, i^2)$, $i = 1, 2, 3$ are three independent normal random variables, then give an expression for t statistic with 2 degrees of freedom using X_1, X_2, X_3.

24. Using statistical table, find the left tailed critical values corresponding to area 0.05 for

(a) Chi-square distribution with 10 degrees of freedom

(b) t distribution with 15 degrees of freedom

25. Let $(X_1, X_2, \ldots X_n)$ be a sequence of independent normal random variables such that $X_i \sim N(\mu, \sigma_i^2)$, $i = 1, 2, \ldots, n$. Define a non-central Chi-square random variable $(X_1, X_2, \ldots X_n)$.

26. If $X \sim F(m, n)$ write down the probability density function of $\frac{1}{X}$.

(8 × 2 = 16 Marks)

SECTION – C

(Answer **any six** Questions. Each question carries **4** marks)

27. Explain sample space, sigma field and probability measure.

28. If $A_1, A_2, \ldots, A_n \ldots$ is a sequence of events in sample space S such that $A_1 \subseteq A_2 \subseteq \ldots \subseteq A_n$ then prove that $P\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n \to \infty} P(A_n)$.

29. Establish weak law of large numbers for a random sample $X_1, X_2, \ldots, X_n$ drawn from a population with mean μ and variance σ^2.

30. Let $\{X_k\}$ be a sequence of independent random variables with values -2^k, 0 and 2^k and probabilities $P(X_k = \pm 2^k) = 2^{-(2k+1)}$; $P(X_k = 0) = 1 - 2^{-2k}$. Examine whether weak law of large numbers holds for the sequence.

31. Let the probability density function of a random variable X be $f(x) = 1;\ 0 < x < 1$. What is the lower bound of $P\left(\left|X - \frac{1}{2}\right| \le 2\sqrt{\frac{1}{12}}\right)$ when one uses the Chebyshev's inequality?

32. Let $X_1, X_2, \ldots, X_n$ be a random sample from a uniform distribution over (0,1). Find the probability density function of r^{th} order statistic $X_{(r)}$.

33. Let $\overline{X}$ be the sample mean of a random sample of size 50 from a normal population with mean 112 and standard deviation 40. Find (a) $P(110 < \overline{X} < 114)$ (b) $P(\overline{X} > 113)$

34. Let X_1, X_2, X_3, X_4 be a random sample from a normal distribution with variance equal to 9 and let $S^2 = \frac{1}{3}\sum_{i=1}^{4}(X_i - \overline{X})^2$. Find k such that $P(S^2 \le k) = 0.05$.

35. Let (X_1, X_2) be a random sample from a distribution with density function $f(x) = e^{-x};\ x > 0$ Find the density function of $Y = \min(X_1, X_2)$.

36. Find the second central moment μ_2 of a t distribution with n degrees of freedom.

37. If $X \sim F(n, n)$ is a F variable with (n, n) degrees of freedom, find the median of the distribution of X.

38. Let $(X_1, X_2, \ldots, X_m)$ and $(Y_1, Y_2, \ldots, Y_n)$ be two independent random samples of sizes m and n respectively from a standard normal population $N(0,1)$. What is the sampling distribution of $W = \dfrac{n\sum_{i=1}^{m} X_i^2}{m\sum_{i=1}^{n} Y_i^2}$. Hence obtain mean of W.

(6 × 4 = 24 Marks)

SECTION – D

(Answer **any two** Questions. Each question carries **15** marks)

39. (a) If X is a continuous random variable with mean μ and variance σ^2, establish Chebyshev's inequality.

(b) If X is a random variable with $E(X)=3$ and $E(X^2)=13$, use Chebyshev's inequality to determine the lower bound for the probability $P(-2< X <8)$.

40. (a) State and prove Lindberg-Levy form of central limit theorem.

(b) If $X_1, X_2, \ldots, X_n$ is a sequence of Bernoulli random variables with probability success p, write down the central limit theorem result.

41. (a) Suppose the mean weight of school children's book bag is 1.74 kilograms with standard deviation 0.22. Find the probability that the mean weight of a sample of 300 book bags will exceed 1.7 kilograms.

(b) Suppose the mean number of days to germination of a variety of seed is 22 with standard deviation 2.3 days. Find the probability that the mean germination time of a sample of 160 seeds will be within 0.5 day of the population mean.

42. If $X_1, X_2, \ldots, X_n$ is a random sample drawn from a population with distribution function $F(x)$ find the distribution function of r^th^ order statistic $X_{(r)}$. If random variables are continuous obtain probability density function of $X_{(r)}$.

43. If $\{X_i\}$ is a sequence of independent standard normal random variables, find moment generating function of $Y = \sum_{i=1}^{n} X_i^2$. Identify the distribution of Y and write down its probability density function.

44. (a) Define t, χ^2 and F statistics and give relationship between each of them.

(b) Obtain r^{th} arbitrary moment μ_r' of F distribution with (m, n) degrees of freedom.

(2 × 15 = 30 Marks)