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Reg. No. : ....

Name : ....

Sixth Semester B.Sc. Degree Examination, April 2022

First Degree Programme under CBCSS

Mathematics

Core Course XII

MM 1644 – ABSTRACT ALGEBRA II

(2014 & 2017 Admission)

Time : 3 Hours Max. Marks : 80

SECTION – I

All the first **ten** questions are compulsory. They carry **one** mark each.

1. Define Kernel of a group homomorphism.

2. Let H be a normal subgroup of G. Prove that $\gamma : G \to G/H$ given by $\gamma(x) = xH$ is a homomorphism with kernel H.

3. Define automorphism of a group.

4. $\mathbb{Z}/3\mathbb{Z}$ is isomorphic to ——————

5. Find the order of the factor group $\mathbb{Z}_6/\langle 2\rangle$

6. Define skew field.

7. Find all units in the ring $\mathbb{Z}_6$.

8. Explain characteristic of a ring.

P.T.O.

9. Give an example of an integral domain that is not a field.

10. Is there exist a field of 7 elements.

(10 × 1 = 10 Marks)

SECTION – II

Answer any **eight** questions from among the questions **11** to **22**. These questions carry **2** marks each.

11. Let S_n be the symmetric group on n letters, and let $\phi : S_n \to \mathbb{Z}_2$ be

    defined by $\phi(\sigma) = \begin{cases} 0 \text{ if } \sigma \text{ is an even permutation,} \\ 1 \text{ if } \sigma \text{ is an odd permutation} \end{cases}$

    Show that ϕ is a homomorphism.

12. State the Fundamental Homomorphism Theorem.

13. How many homomorphisms are there of $\mathbb{Z}$ onto $\mathbb{Z}$?

14. Prove that every subgroup H of an abelian group G is normal.

15. Find the order of the element $5 + \langle 4 \rangle$ in the factor group $\mathbb{Z}_{12} / \langle 4 \rangle$.

16. Define ring.

17. Find the characteristic of the ring $\mathbb{Z}_3 \times 3\mathbb{Z}$.

18. State Little Theorem of Fermat.

19. Let m be a positive integer and let $a \in \mathbb{Z}_m$ be relatively prime to m. Prove that for each $b \in \mathbb{Z}_m$, the equation $ax = b$ has a unique solution in $\mathbb{Z}_m$.

20. Find all solutions of the congruence $12x \equiv 27 (\text{mod } 18)$.

21. Distinguish between ideal and subring.

22. Explain factor ring.

(8 × 2 = 16 Marks)

SECTION – III

Answer any **six** questions from among the questions **23** to **31**. These questions carry **4** marks each.

23. Prove that a factor group of a cyclic group is cyclic.

24. Compute the factor group $(\mathbb{Z}_4 \times \mathbb{Z}_6)/\langle(0, 2)\rangle$.

25. Prove that the converse of Lagrange theorem is not true.

26. If R is a ring with additive identity 0, then prove that for any $a, b \in R$
    (a) $0a = a0 = 0$
    (b) $a(-b) = (-a)b = -(ab)$
    (c) $(-a)(-b) = ab$.

27. Solve the equation $x^2 - 5x + 6 = 0$ in $\mathbb{Z}_{12}$.

28. Prove that the cancellation laws hold in a ring R if and only if R has no divisors of 0.

29. Show that for every integer n, the number $n^{33} - n$ is divisible by 15.

30. State and prove Euler's theorem.

31. Find all solutions of the congruence $15x \equiv 27 \pmod{18}$.

(6 × 4 = 24 Marks)

SECTION – IV

Answer any **two** questions from among the questions 32 to 35. These questions carry 15 marks each.

32. (a) Let ϕ be a homomorphism of a group G into a group G'. Prove that
    (i) If e is the identity element in G, then $\phi(e)$ is the identity element e' in G'.
    (ii) If $a \in G$, then $\phi(a^{-1}) = \phi(a)^{-1}$.
    (iii) If H is a subgroup of G, then $\phi[H]$ is a subgroup of G'.
    (iv) If K' is a subgroup of $G' \cap \phi[G]$, then $\phi^{-1}[K']$ is a subgroup of G.

    (b) A group homomorphism $\phi : G \to G'$ is a one-to-one map if and only if $Ker(\phi) = \{e\}$.

33. (a) Let H be a subgroup of a group G. Prove that left coset multiplication is well defined by the equation.

$$(aH)(bH) = (ab)H$$

If and only if H is a normal subgroup of G.

(b) Prove that the following are three equivalent conditions for a subgroup H of a group G to be a normal subgroup of G.

(i) $gHg^{-1} \in H$ for all $g \in G$ and $h \in H$.

(ii) $gHg^{-1} = H$ for all $g \in G$.

(iii) $GH = Hg$ for all $g \in G$.

34. (a) In the ring $\mathbb{Z}_n$, prove that the divisors of 0 are precisely those non zero elements that are not relatively prime to n.

(b) Prove that every field F is an integral domain.

(c) Prove that every finite integral domain is a field.

35. (a) Prove that the set $\mathbb{G}_n$ of nonzero elements of $\mathbb{Z}_n$ that are not 0 divisors forms a group under multiplication modulo n.

(b) Let H be a subring of the ring R. Prove that multiplication of additive cosets of H is well defined by the equation.

$$(a+H)(b+H) = ab + H$$

if and only if $ah \in H$ and $hb \in H$ for all $a, b \in R$ and $h \in H$.

(2 × 15 = 30 Marks)