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**Reg. No. : ..............................**

**Name : ..................................**

**Sixth Semester B.Sc. Degree Examination, April 2022**

**First Degree Programme under CBCSS**

**Mathematics**

**Core Course – XI**

**MM 1643 : ABSTRACT ALGEBRA – RING THEORY**

**(2018 & 2019 Admission)**

Time : 3 Hours Max. Marks : 80

SECTION – A

Answer **all** questions. Each carries **1** mark.

1. Which are the units of $Z$?

2. Define a zero divisor.

3. Give an example of a non commutative ring.

4. Define an ideal of a ring.

5. State the first isomorphism theorem for rings.

6. Is the ring $2z$ is isomorphic to the ring $3z$?

7. Define the content of a nonzero polynomial.

8. Define an irreducible element in an integral domain.

9. Find the norm of $a+b\sqrt{d}$ , where $a$ and $b$ are integers.

10. Define a Noetherian domain.

**(10 × 1 = 10 Marks)**

P.T.O.

## SECTION – B

Answer **any eight** questions. Each carries **2** marks.

11. If $a$ is an element in a ring $R$, then prove that $a0 = 0a = 0$

12. Prove that if a ring element has a multiplicative inverse, then it is unique.

13. Let $a$, $b$ and $c$ belong to an integral domain. Prove that if $a \neq 0$ and $ab = ac$, then $b = c$.

14. Show that 0 is the only nilpotent element in an integral domain.

15. Prove that the ideal $\langle x^2 + 1 \rangle$ is maximal in $R[x]$.

16. Prove that the only ideals of a field $F$ are $\{0\}$ and $F$ itself.

17. Determine all ring homomorphisms from $z$ to $z$.

18. Find a polynomial with integer coefficients that has $\frac{1}{2}$ and $\frac{-1}{3}$ are zeros.

19. Give an example of a field that properly contains the field of complex numbers $c$.

20. Let $F$ be a field and let $a$ be a non zero element of $F$. Show that if $f(x+a)$ is irreducible over $F$, then $f(x)$ is irreducible over $F$.

21. Construct a field of order 25.

22. Find the kernel of the ring homomorphism $\varphi$ from $R[x]$ onto $c$ given by $f(x) \to f(i)$.

23. Prove that in an integral domain, every prime is irreducible.

24. Show that $z[x]$ is not a principal ideal domain.

25. Find $q$ and $r$ in $z[i]$ such that $3 - 4i = (2 + 5i)q + r$ and $d(r) < d(2 + 5i)$.

26. Show that for any non trivial ideal $I$ of $z[i]$, $\frac{z[i]}{I}$ is finite.

**(8 × 2 = 16 Marks)**

## SECTION – C

Answer **any six** questions. Each carries **4** marks.

27. Let $R$ be a ring. Prove that $a^2 - b^2 = (a+b)\ (a-b) \forall\ a, b \in R$ if and only if $R$ is commutative.

28. Let $R$ be a ring with unity 1. Show that if 1 has order $n$ under addition, then the characteristic of $R$ is $n$.

29. Show that a finite commutative ring with no zero divisors and at least two elements has a unity.

30. Find all solutions of the equation $x^3 - 2x^2 - 3 = 0$ in $z_{12}$.

31. Let $\varphi$ be a ring homomorphism from a ring $R$ to a ring $S$. Let $A$ be a subring of $R$. Prove that if $A$ is an ideal and $\varphi$ is onto $S$, then $\varphi(A)$ is an ideal.

32. Let $F$ be a field. If $f(x) \in F[x]$ and $\deg f(x)$ is 2 or 3, then prove that $f(x)$ is reducible over $F$ if and only if $f(x)$ has a zero in $F$.

33. Find the quotient and remainder upon dividing $f(x) = 3x^4 + x^3 + 2x^2 + 1$ by $g(x) = x^2 + 4x + 2$, where $f(x)$ and $g(x)$ belong to $z_5(x)$.

34. Check whether $f(x) = 21x^3 - 3x^2 + 2x + 9$ is an irreducible polynomial over $Q$.

35. Show that 7 is irreducible in the ring $z[\sqrt{5}]$.

36. Prove that every Euclidean Domain is an Principal Ideal Domain.

37. Prove that in a PID, any strictly increasing chain of ideals $I_2 \subset I_3 \subset \ldots$ must be finite in length.

38. Let $D$ be a Euclidean Domain with measure $d$. Show that if $a$ and $b$ are associates in $D$, then $d(a) = d(b)$.

**(6 × 4 = 24 Marks)**

## SECTION – D

Answer **any two** questions. Each carries **15** marks.

39. (a) Let $d$ be an integer. Prove that $z(\sqrt{d})-\{a+b\sqrt{d}\mid a, b\in z\}$ is an integral domain.

    (b) Define center of a ring $R$. Prove that the center of a ring is a subring.

40. Let $R$ be a ring and let $A$ be a subring of $R$. Prove that the set of cosets $\{r+A\mid r\in R\}$ is a ring under the operations $(s+A)+(t+A)=s+t+A$ and $(s+A)\,(t+A)=st+A$ if and only if $A$ is an ideal of $R$.

41. State and prove division algorithm for $F[x]$.

42. Let $F$ be a field. Prove that $F[x]$ is a principal ideal domain.

43. Let $F$ be a field and let $p(x)\in F[x]$. Then prove that $\langle p(x)\rangle$ is a maximal ideal in $F[x]$ if and only if $p(x)$ is irreducible over $F$.

44. Show that the ring $Z[\sqrt{-5}]$ is not a Unique Factorisation Domain.

**(2 × 15 = 30 Marks)**

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