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Reg. No. : ....

Name : ....

Third Semester M.Sc. Degree Examination, February 2021

Mathematics

MM 231 : COMPLEX ANALYSIS — I

(2005 Admission Onwards)

Time : 3 Hours Max. Marks : 75

1. Answer either PART-A or PART-B of each question.

2. All questions carry equal marks.

I. (A) (a) Let $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$ have radius of convergence $R > 0$. Show that

(i) $R = \lim \left| \dfrac{a_n}{a_{n+1}} \right|$, if the limit exists.

(ii) the series $\sum_{n=1}^{\infty} n a_n (z-a)^{n-1}$ has the radius of convergence R.

(b) Find the radius of convergence of the series $\sum_{n=0}^{\infty} k^n \Sigma^n$, where k is a positive integer.

(12 + 3 = 15 Marks)

OR

P.T.O.

(B) (a) If G is open and connected and $f : G \to C$ is differentiable with $f'(z) = 0$ for all z in G, prove that f is constant.

(b) Let u and v be real-valued functions defined on a region G and suppose that u and v have continuous partial derivatives. Then prove that $f : G \to \mathbb{C}$ defined by $f(z) = u(z) + iv(z)$ is analytic if and only if u and v satisfy the Cauchy-Riemann equations. **(5 + 10 = 15)**

II. (A) (a) Let f be analytic in $B(a\,;R)$ and suppose $|f(z)| \le M$ for all z in $B(a\,;R)$. Obtain Cauchy's estimate for $\left|f^{(n)}(a)\right|$.

(b) State and prove

(i) Liouville's theorem.

(ii) Fundamental theorem of algebra. **(4 + 11 = 15)**

OR

(B) Let $\gamma : [0, 1] \to \mathbb{C}$ be a closed rectifiable curve and $a \notin \{\gamma\}$

(a) Prove that $\frac{1}{2\pi i}\int_{\gamma} \frac{d^{z}}{z-a}$ is an integer.

(b) Define $n(\gamma; a)$, the index of γ with respect to a.

(c) Prove that $n(\gamma : a)$ is constant for a belonging to a component of $G = \mathbb{C} - \{\gamma\}$.

(d) Prove that $n(\gamma\,; a) = 0$ for a belonging to the unbounded component of G. **(8 + 2 + 3 + 2 = 15)**

III. (A) (a) State and prove Cauchy's integral formula (first version).

(b) If G is simply connected and $f : G \to \mathbb{C}$ is analytic in G, show that f has a primitive in G. **(8 + 7 = 15)**

OR

(B) (a) Let G be a region and let f be an analytic function on G, with zeros $a_1,....,a_m$ (repeated according to multiplicity) . If γ is a closed rectifiable curve in G, which does not pass through any point a_k and if $\gamma \simeq 0$, show that

$$\frac{1}{2\pi i}\int_{\gamma}\frac{f'(z)}{f(z)}\,dz = \sum_{k=1}^{m} n(\gamma;\, a_n).$$

(b) Calculate $\int_{\gamma}\frac{(2z+1)}{z^2+z+1}dz$, where γ is the circle $|z| = 2$.

(c) Suppose f is analytic in $B(a\,;R)$ and let $\alpha = f(a)$. If $f(z)-\alpha$ has a zero of order m at $z = a$, prove there is $a_n \in > 0$ and $\delta > 0$ such that for $|\xi - \alpha| < \delta$, the equation $f(z) = \xi$ has exactly m simple roots in $B(a\,;\in)$. **(5 + 3 + 7 = 15)**

IV. (A) State and prove

(a) Residue theorem

(b) Rouche's theorem. **(8 + 7 = 15)**

OR

(B) Show that

$$\int_{-\infty}^{\infty}\frac{x^2}{1+x^4}dx = \frac{\pi}{\sqrt{2}}.$$ **(15)**

V. (A) State and prove

(a) Maximum modulus theorem. (second version)

(b) Schwarz's lemma. **(7 + 8 = 15)**

OR

(B) (a) Let z_1, z_2, z_3, z_4 be four distinct points in $\mathbb{C}_\infty$. Prove that (z_1, z_2, z_3, z_4) is a real number if and only if all four points lie on a circle.

(b) Let Γ be a circle through points z_2, z_3, z_4. When do you say that the points z and z^* in $\mathbb{C}_\infty$ are symmetric with respect to Γ?

(c) Obtain the symmetry principle. **(5 + 2 + 8 = 15)**