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N – 5376

Reg. No. : ....

Name : ....

First Semester M.Sc. Degree Examination, May 2022

Mathematics

MM 214 — TOPOLOGY – I

(2020 Admission Onwards)

Time : 3 Hours

Max. Marks : 75

PART – A

Answer any **five** questions. **Each** question carries **3** marks.

1. Compute the distance from $a=(-2,1)$ and b = (3, 4) in $\mathbb{R}^2$ with respect to the following three metrics.

   (a) usual metric

   (b) taxicab metric

   (c) max metric

2. In the plane $\mathbb{R}^2$ describe the open ball with center 0 and radius 1 with respect to the following metrics.

   (a) Usual metric

   (b) Taxicab metric

3. Define metrically equivalent spaces and show that metric equivalence is an equivalence relation.

P.T.O.

4. Define a nowhere dense subset of a metric space and give two examples of nowhere dense sets in $\mathbb{R}^2$.

5. Determine the smallest and largest topologies on any set X.

6. For a subset A of a topological space X, prove that A is open if and only if $bdy(A) \subset (X - A)$.

7. If $\mathbb{R}$ is the space of real numbers with finite complement topology, is $\mathbb{R}^2$ connected or disconnected? Justify your answer.

8. Find the one-point compactification of the real line $\mathbb{R}$.

(5 × 3 = 15 Marks)

PART – B

Answer **all** questions. **Each** question carries **12** marks.

9. (A) (a) State and prove the Minkowski inequality. **6**

(b) With usual notations prove that $\mathbb{C}[a, b]$ is a metric spare with ρ defined by $\rho(f,g) = \int_a^b |f(x) - g(x)| dx$. **6**

OR

(B) (a) Prove that a sequence in a metric space cannot converge to more than one limit. **6**

(b) If A is a subset of a metric space X, prove that $\overline{A}$ is a closed set and is a subset of every closed set containing A. **6**

10. (A) Prove that the following statements are equivalent for a function f from metric space (X, d) to metric space (Y, d') :

(i) f is continuous

(ii) For each sequence (x_n) converging to a in $X, (f(x_n))$ converges to $f(a)$

(iii) For each open set O in $Y, f^{-1}(O)$ is open in X

(iv) For each closed set C in $Y, f^{-1}(C)$ is closed in X. **12**

OR

(B) (a) Show that every metric space is topologically equivalent to a bounded metric space. **6**

(b) Stare and prove the Cantor's Intersection theorem. **6**

11. (A) (a) Prove that the Hilbert space is separable. **6**

(b) Prove hat a family $\mathcal{B}$ of subsets of a set X is a basis for some topology for X if and only if both of the following conditions hold :

(i) The union of the members of $\mathcal{B}$ is X

(ii) For each B_1, B_2 in $\mathcal{B}$ and $x \in_1 B_1 \cap B_2$ there is a member B_x of $\mathcal{B}$ such that $x \in B_x \subset B_1 \cap B_2$. **6**

OR

(B) (a) Prove that the following are topological properties :

(i) Separability

(ii) First Countability

(iii) Second Countability **6**

(b) Define the Zariski topology. Show that $\mathbb{R}^n$ with Zariski topology is not Hausdorff. **6**

12. (A) Prove that the following statements are equivalent for a topological space X

(i) X is disconnected

(ii) X is the union of two disjoint, non-empty closed sets

(iii) X is the union of two separated sets

(iv) There is a continuous function f from X onto a discrete two point space $\{a, b\}$

(v) X has a proper subset A which is both open and closed.

(vi) X has a proper subset A such that $\overline{A} \cap \overline{X} = \phi$. **12**

OR

(B) (a) Prove that connected subsets of $\mathbb{R}$ are precisely the intervals. **6**

(b) Prove that every closed and bounded interval has the fixed point property. **6**

13. (A) (a) Prove that a spare X is compact every if and only if every family of closed sets in X with the finite intersection property has nonempty intersection. **6**

(b) Let (X, d) be a compact metric space, (Y, d') a metric space and $f : X \to Y$ a continuous function. Prove that f is uniformly continuous. **6**

OR

(B) Prove that the following are equivalent for a subset A of $\mathbb{R}^n$

(i) A is compact.

(ii) A has the Bolzano-Weierstrass property.

(iii) A is countably compact.

(iv) A is closed and bounded. **12**

(5 × 12 = 60 Marks)