(Pages : 4) **P – 6031**

Reg. No. : ....

Name : ....

Third Semester M.Sc. Degree Examination, January 2023

Mathematics

MM 231 — COMPLEX ANALYSIS

(2020 Admission Onwards)

Time : 3 Hours Max. Marks : 75

PART – A

Answer **any five** questions. Each question carries **3** marks.

1. Prove that every convergent series in $\mathbb{C}$ is absolutely convergent.

2. State Cauchy's estiamte.

3. Define index or winding number $n(\gamma, a)$. Show that $n(\gamma, a) = -n(-\gamma, a)$ for every $a \notin \{\gamma\}$.

4. Give an example of a closed rectifiable curve γ in a region G such that $n(\gamma; w) = 0$ for all $w \in \mathbb{C} - G$ where as γ is not homotopic to a constant curve.

5. State Cauchy's theorem and Goursat's theorem.

6. Evaluate $\int_0^\infty \frac{dx}{(1+x^2)}$.

P.T.O.

7. Let $z = a$ be an isolated singularity of f and let $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n$ be its Laurent expansion. Prove that $z = a$ is a removable singularity if and only if $a_n = 0$ and $n \leq -1$.

8. Prove that a Mobius transformation takes circles into circles.

(5 × 3 = 15 Marks)

PART – B

Answer **all** questions. Each question carries **12** marks.

9. A. (a) For a given power series $\sum_{n=0}^{\infty} a_n (z-a)^n$, let R be defined by $0 \leq R \leq \infty$ and $\frac{1}{R} = \lim\sup |a_n|^{1/n}$. If $|z-a| < R$, prove that the series $\sum_{n=0}^{\infty} a_n (z-a)^n$ converges absolutely. **4**

(b) Let $G \subset \mathbb{C}$ be open and let γ be a rectifiable path in G with initial and end points α and β respectively. If $f : G \to \mathbb{C}$ is a continuous function with a primitive $F : G \to \mathbb{C}$, show that $\int_{\gamma} f = F(\beta) - F(\alpha)$. **8**

OR

B. (a) If G is open and connected and $f : G \to \mathbb{C}$ is differentiable with $f'(z) = 0$ for all $\alpha \in G$, prove that f is constant. **4**

(b) Let G be either the whole plane $\mathbb{C}$ or some open disk. If $u : G \to \mathbb{R}$ is a harmonic function, show that u has a harmonic conjugate. **4**

(c) If γ is piecewise smooth and $f : [a, b] \to \mathbb{C}$ is continuous, show that $\int_a^b f d\gamma = \int_a^b f(t) \gamma'(t) dt$. **4**

10. A. (a) Let f be analytic in $B(a;R)$ and $a_n = \frac{1}{n!} f^{(n)}(a)$. Show that $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$ for $|z-a| < R$ and this series has radius of convergence $\geq R$. **6**

(b) Let γ be a closed rectifiable curve in $\mathbb{C}$. Prove that $n(\gamma : a)$ is constant for a be longing to a component of $G = \mathbb{C} - \{\gamma\}$. Further prove that $n(\gamma, a) = 0$ for a belonging to the unbounded component of G. **6**

OR

B. (a) Let $f : G \to \mathbb{C}$ be analytic and $\overline{B}(a;r) \subset G (r > 0)$. If $\gamma(t) = a + re^{it}$, $0 \leq t \leq 2\pi$, prove that, for $|z-a| < r$,

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w-z} dw.$$ **6**

(b) State and prove the Fundamental Theorem of Algebra. **6**

11. A. (a) Let G be a region and let $f : G \to \mathbb{C}$ be a continuous function such that $\int_T f = 0$ for every triangular path T in G. Prove that the function f is analytic in G. **6**

(b) State and prove Independence Path Theorem. **6**

OR

B. If γ_0 and γ_1 are two closed rectifiable curves in G and $\gamma_0 \sim \gamma_1$, prove that $\int_{\gamma_0} f = \int_{\gamma_1} f$ for every function f analytic in G. **12**

12. A. (a) State and prove Casorati-Weierstrass theorem. **6**

(b) State and prove Rouche's theorem. **6**

OR

B. (a) Prove that $\int_{-\infty}^{\infty} \frac{x^2}{1+x^4} dx = \frac{\pi}{\sqrt{2}}$. **6**

(b) State and prove the Argument Principle. **6**

13. A. (a) State and explain the Symmetry Principle through an example. **6**

(b) Prove that the cross ratio of four distinct points in $\mathbb{C}_\infty$ is real if and only if all four points lie on a circle. **6**

OR

B. (a) Prove that Mobius transformation takes circles into circles. **6**

(b) State and prove Schwarz's Lemma. **6**

(5 × 12 = 60 Marks)

(Pages : 4) **P – 6032**

Reg. No. : ....

Name : ....

Third Semester M.Sc. Degree Examination, January 2023

Mathematics

MM 232 : FUNCTIONAL ANALYSIS – I

(2020 Admission Onwards)

Time : 3 Hours Max. Marks : 75

PART – A

Answer any **five** questions. Each question carries **3** marks.

1. Let X be a normed linear space. For for $x \in X$ and $r > 0$, prove that the open ball $U(x, r) = \{y \in X : \|x - y\| < r\}$ and its closure $\overline{U}(x, r)$ are convex subsets of X.

2. If a normed linear space X is finite dimensional, prove that X is complete.

3. Define support functional and support hyperplane. Give an example.

4. Define summable and absolutely summable series in a Banach space.

5. Show that $F \in BL(X, y)$ is bounded if and only if $y' \circ F \in X'$ for every for every $y' \in Y'$.

6. State Newton-Cotes formulae.

7. Define resolvent set of $A \in BL(X, Y)$.

8. Let X and Y be normed linear spaces and $F \in BL(X, Y)$. Show that F' is one to one if and only if $R(F)$ is dense in Y.

(5 $\times$ 3 = 15 Marks)

PART – B

Answer **all** questions. Each question carries **12** marks.

9. (A) (a) If $f : X \to K$ is a linear functional, prove that either $f \equiv 0$ or f maps open sets in X onto open sets in K. **4**

(b) If Y is a closed subspace of a normed linear space X, prove that the quotient space X/Y is a normed space. Also prove that a sequence $(x_n + Y)$ in X/Y converges to $x + y \in X/Y$ if and only if there exists a sequence (y_n) in Y such that $(x_n + y_n)$ converges to x in X. **5**

(c) Let X be a normed linear space which is linearly homeomorphic to a complete normed linear space Y. Show that X is also complete. **3**

OR

(B) (a) Let X be a normed linear space over K and let $F : X \to K$ be a linear map. Prove that F is continuous at 0 if and only if $Z(F)$ is closed in X. **6**

(b) Let X be a normed linear space and Y a closed subspace of X with $Y \neq X$. Prove that there exists $x_r \in X$ such that $\|x_r\| = 1$ and $r \leq d(x, Y) \leq 1$ for every $0 < r < 1$. **6**

10. (A) (a) Let X be a normal linear space. Prove that every bounded linear functional on every subspace of X has a unique norm-preserving linear extension to X if and only if X' is strictly convex. **6**

(b) Let $X \neq \{0\}$ and Y be two normed linear spaces. Prove that $BL(X, Y)$ is Banach if and only if Y is Banach space. **6**

OR

(B) (a) Let X be a normed linear space over $\mathbb{R}$, E a non-empty open convex subset of X, and Y a subspace of X such that $E \cap Y = \phi$. Prove that there exists a closed hyperspace H in X such that $Y \subset H$ and $E \cap H = \phi$. **5**

(b) Let X be a Banach space. Show that X cannot have a denumerable basis. **4**

(c) Let Y be a subspace of X and $a \in X$, prove that $a \in \overline{Y}$ if and only if $f(a) = 0$ whenever $f \in X'$ and $f \equiv 0$ on Y. **3**

11. (A) (a) Let X be a normed linear space and $E \subset X$. Prove that E is bounded in X if and only if $x'(E)$ is bounded in K for every $x' \in X'$. **6**

(b) State and prove the closed graph theorem. **6**

OR

(B) (a) Let X be Banach space over $\mathbb{C}$, D an open subset of $\mathbb{C}$ and $F : D \to X$. Prove that F is analytic on D if and only if $x' \circ F$ is analytic on D for every $x' \in X'$. **6**

(b) Give an example to show that the completeness assumption in the closed graph theorem cannot be omitted. **6**

12. (A) (a) Let X and Y be two normed linear spaces. State and prove Neumann expansion for $A \in BL(X, Y)$. **6**

(b) Find $e(S)$ and $a(S)$ for the right shift operator S on a Banach sequence space X. **6**

OR

(B) (a) Let X be a linear space and $A \in BL(X)$ be of finite rank. Prove that $s(A) = e(A)$. **6**

(b) Let X be a linear space and $P \in BL(X)$ be a projection. Prove that

$$e(P) = s(P) = \begin{cases} \{0\} & \text{if } P = 0 \\ \{1\} & \text{if } P = I \\ \{0, 1\} & \text{if } 0 \neq P \neq I \end{cases}$$ **6**

13. (A) Let X and Y be Banach spaces and $F \in BL(X, Y)$. Show that F is onto if and only if F' is bounded above. **12**

OR

(B) (a) State and prove a criterion to determine the weak convergence of a sequence in a normal linear space X. **6**

(b) Let X and Y be normed linear spaces and $X \neq \{0\}$. Prove that CL(X, Y) is Banach if and only if Y is Banach. **6**

(5 × 12 = 60 Marks)

(Pages : 3) **P – 6043**

Reg. No. : ....

Name : ....

Third Semester M.Sc. Degree Examination, January 2023

Mathematics

Elective II

MM 234 : GRAPH THEORY

(2020 Admission Onwards)

Time : 3 Hours Max. Marks : 75

SECTION – A

Answer **any five** questions. **Each** question carries **3** marks.

1. Show that a vertex V of a connected graph G is a Cut-vertex of G if and only if there exist vertices u and w distinct from v such that v lies on every u-w path of G.

2. If G is a graph of order n and size $m \geq n-1$, then show that $K(G) \leq \left[\frac{2m}{n}\right]$

3. If G is a Hamiltonian graph, then show that for every nonempty proper set S of vertices of G, $K(G-S) \leq |S|$

4. Determine the value of $\alpha(G)$, $\beta(G)$, $\alpha_1(\beta)$ and $\beta_1(G)$ for $K_1 + 2K_3$.

5. Show that every r-regular bipartite graph, $r \geq 1$, is 1-factorable.

6. Let G be a graph of odd order n and size m. If $m > \frac{(n-1)\Delta(G)}{2}$, then show that $\chi_1(G) = 1 + \Delta(G)$.

P.T.O.

7. For every two adjacent vertices u and v in a connected graph, prove that $|e(u) - e(v)| \leq 1$.

8. Show that no cut – vertex of a connected graph G is a boundary vertex of G.

(5 $\times$ 3 = 15 Marks)

SECTION – B

Answer **all** questions. **Each** question carries 1**2** marks.

9. (A) (i) Show that $K(H_{3,n}) = 3$

(ii) Prove that isomorphism is an equivalence relation on the set of all graphs.

(B) (i) For every graph G, show that $K(G) \leq \lambda(G) \leq \delta(G)$.

(ii) If G and H are isomorphic graph, then show that the degrees of the vertices of G are the same as the degrees of the vertices of H.

10. (A) (i) Prove that the Petersen graph is non-Hamiltonian.

(ii) Let G be a graph of order $n \geq 3$. If for every integer j with $1 \leq \int < \frac{n}{2}$, then number of vertices of G with degree almost j is loss then j, then show that G is Hamiltonian.

(B) (i) Show that a connected graph G contains an Eulerian trail iff exactly two vertices of G have odd degree. Further more, each Eulerian trail of G begins at one of these odd vertices and ends at the other.

(ii) For every connected graph G, show that $h^*(G) = h(G)$, where $h(G)$ is the length of a Hamiltonian walk in G and $h^*(G)$ is the Hamiltonian number of G.

11. (A) (i) Show that a digraph D is strong if and only if D contains a closed spanning walk.

(ii) Show that for every graph G of order n containing no isolated vertices, $\alpha_1(G)+\beta_1(G)=n$.

(B) (i) If u is a vertex of maximum out degree in a tournament T, then show that $d(u,v)\leq 2$ for every vertex v of T.

(ii) State and prove the Petersen's theorem.

12. (A) (i) If G is a non empty bipartite graph, then show that $\chi_1(G)=\Delta(G)$.

(ii) Prove that $r(K_3,K_4)=9$

(B) (i) For every graph G, show that $\chi(G)\leq 1+\max\{\delta(H)\}$, where the maximum is taken over all induced sub graphs H of G.

(ii) Show that every graph of order $n\geq 3$ and size atleast $\binom{n-1}{2}+2$ is Hamiltonian.

13. (A) (i) Show that the center of every connected graph G is a subgraph of some block of G.

(ii) Show that a connected graph G of order n has locating number 1 iff $G\cong P_n$.

(B) (i) Show that non trivial graph G is the eccentric sub graph of some graph if and only if every vertex of G has eccentricity 1 or no vertex of G has eccentricity 1.

(ii) Prove that for every nontrivial connected graph G, $rad_D(G)\leq diam_D(D)\leq 2rad_D(G)$.

(5 $\times$ 12 = 60 Marks)

(Pages : 4) **P – 6035**

Reg. No. : ....

Name : ....

Third Semester M.Sc. Degree Examination, January 2023

Mathematics

Elective I

MM 233 – OPERATIONS RESEARCH

(2020 Admission Onwards)

Time : 3 Hours Max. Marks : 75

SECTION – A

Answer **any five** questions. **Each** question carries **3** marks.

1. Define
   (a) Basic Solution
   (b) Basic feasible solution
   (c) Optimum basic feasible solution.

2. What is meant by standard form of an LPP?

3. Using Vogel's approximation method, find an initial basic feasible solution of the transportation problem:

	D_1	D_2	D_3	D_4	Supply
S_1	1	2	1	4	30
S_2	3	3	2	1	50
S_3	4	2	5	9	20
Demand	20	40	30	10	

4. What is an assignment problem? Give the mathematical formulation of an assignment problem.

P.T.O.

5. Explain the following terms in PERT/CPM.

(a) Total activity time

(b) Event Slack

(c) Critical Path

6. What do you mean by non-linear programming problem?

Define Lagrangian function for the non-linear programming problem:

Minimize $f(X)$ subject to $g_i(X) \leq 0, i = 1, 2, ... n$.

7. Write the Kuhn-Tucker conditions for:

Minimize $f = (x_1 - 2)^2 + x_2^2$ subject to $x_1^2 + x_2 - 1 \leq 0, x_1, x_2 \geq 0$.

8. Explain the computational economy in dynamic programming.

(5 $\times$ 3 = 15 Marks)

SECTION – B

Answer **all** questions. **Each** question carries **12** marks.

9. (a) Use two phase simplex method to solve:

Minimize $z = x_1 + x_2$

Subject to $2x_1 + x_2 \geq 4$; $x_1 + 7x_2 \geq 7$; $x_1, x_2 \geq 0$

OR

(b) Solve the following LPP by Big-M method.

Maximise $Z = x_1 + 2x_2 + 3x_3 - x_4$

Subject to

$x_1 + 2x_2 + 3x_3 = 15$; $2x_1 + x_2 + 5x_3 = 20$; $x_1 + 2x_2 + x_3 + x_4 = 10$; $x_1, x_2, x_3, x_4 \geq 0$.

10. (a) Solve the following transportation problem:

	P	Q	R	S	Supply
A	6	3	5	4	22
B	5	9	2	7	15
C	5	7	8	6	8
Demand	7	12	17	9	45

OR

(b) A department company has five employees with five jobs to be performed. The time in hours that each man takes to perform each job is given in the following table:

Employees

		I	II	II	IV	V
	A	10	5	13	15	16
Jobs	B	3	9	18	13	6
	C	10	7	2	2	2
	D	7	11	9	7	12
	E	7	9	10	4	12

How should the jobs be allocated, one per employee, so as to minimize the total man-hours?

11. (a) An architect has been awarded a contract to prepare plans for an urban renewal project. The job consists of the following activities and their estimated times:

Activity	Description	Immediate Predecessors	Time (Days)
A	Prepare preliminary sketches	–	2
B	Outline specifications	–	1
C	Prepare drawings	A	3
D	Write specifications	A, B	2
E	Run off prints	C, D	1
F	Have specification	B, D	3
G	Assemble bid packages	E, F	1

(i) Draw the network diagram of activities for the project.

(ii) Identify the critical path. What is its length?

(iii) Find the total float and free float for each activity.

OR

(b) A research and development department is developing a new power supply for a console television set, It has broken the job down into the following:

Job	Description	Immediate Predecessors	Time (Days)
A	Determine output voltage	–	5
B	Determine whether to use solid state rectifiers	A	7
C	Choose rectifier	B	2

Job	Description	Immediate	Time
D	Choose filters	B	3
E	Choose transformer	C	1
F	Choose chassis	D	2
G	Choose rectifier mounting	C	1
H	Layout chassis	E, F	3
I	Build and test	G, H	10

(i) Draw the network diagram of activities involved in the project and indicate the critical path.

(ii) What is the minimum completion time for the project?

12. (a) Minimize $f(X) = -x_1 - x_2 - x_3 + \frac{1}{2}\left(x_1^2 + x_2^2 + x_3^2\right)$

Subject to

$$g_1(X) = x_1 + x_2 + x_3 - 1 \le 0$$

$$g_2(X) = 4x_1 + 2x_2 - \frac{7}{3} \le 0$$

$$x_1, x_2, x_3 \ge 0$$

OR

(b) Maximize $2x_1 - x_1^2 - x_2$

Subject to $2x_1 + 3x_2 \le 6, 2x_1 + x_2 \le 4, x_1, x_2 \ge 0$.

13. (a) Determine max $\left(u_1^2 + u_2^2 + u_3^2\right)$ subject to $u_1 u_2 u_3 \le 6$ where $u_1 u_2 u_3 \ge 0$.

OR

(b) Show that in a serial two-stage minimization or maximation problem if.

(i) the objective function ϕ_2, is a separable function of stage returns $f_1(X_1, U_1)$ and $f_2(X_2, U_2)$.

(ii) ϕ_2 is monotonic nondecreasing function of f_1 for every feasible value of f_2, then the problem is decomposable.

(5 $\times$ 12 = 60 Marks)