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Reg. No. : ....

Name : ....

Second Semester M.Sc. Degree Examination, September 2022

Mathematics

MM 221 – ABSTRACT ALGEBRA

(2020 Admission Onwards)

Time : 3 Hours Max. Marks : 75

SECTION – A

Answer **any five** questions. **Each** question carries **3** marks.

1. Let $G = U(16)$, $H = \{1, 15\}$ and $K = \{1, 9\}$. Are H and K isomorphic? Are G/H and G/K isomorphic?

2. Prove that a group of order 105 contains a subgroup of order 35.

3. Express $x^8 - x$ as a product of irreducible polynomials over $\mathbb{Z}_2$.

4. Construct a field of order 9.

5. Find $\Phi_{12}(x)$.

6. If a and b are constructible numbers, give a geometric proof that $a + b$ is constructible.

7. Show, by an example, that if the order of a finite abelian group is divisible by 4, the group need not have a cyclic subgroup of order 4.

8. Find the minimal polynomial for $1 + \sqrt[3]{2} + \sqrt[3]{4}$ over $\mathbb{Q}$.

(5 × 3 = 15 Marks)

P.T.O.

SECTION – B

Answer **all** questions. **Each** question carries **12** marks.

9. (A) (a) Let G be an abelian group of prime-power order and let a be an element of maximal order in G. Prove that G is the internal direct product of $\langle a \rangle \times K$ for some subgroup K in G. **9**

(b) Show, by example, that in a factor group G/H it can happen that $aH = bH$ but $|a| \neq |b|$. **3**

OR

(B) (a) Prove that the order of an element of a direct product of a finite number of finite groups is the least multiple of the order of the components of the element. **7**

(b) Express $U(165)$ as an internal direct product of proper subgroups in two different ways. **5**

10. (A) (a) Prove that any two Sylow p-subgroups of a finite group G are conjugate. **7**

(b) Prove that a group of order 175 is abelian. **5**

OR

(B) (a) Suppose that G is a group of order 60 and G has a normal subgroup N of order 2. Prove that G has a cyclic subgroup of order 30. **6**

(b) Prove that if G is a finite group and H is a proper normal subgroup of largest order, then G/H is simple. **6**

11. (A) (a) Prove that a finite extension of a finite extension is finite. **8**

(b) Find the splitting field for $x^3 + x + 1$ over $\mathbb{Z}_2$. **4**

OR

(B) (a) Let $f(x)$ be an irreducible polynomial over a field F and let E be a splitting field of $f(x)$ over F. Prove that all the zeros of $f(x)$ in E have the same multiplicity. **7**

(b) Find the degree and a basis of the splitting field of $x^6 + x^3 + 1$ over $\mathbb{Q}$. **5**

12. (A) (a) Prove that the maximum degree of any irreducible factor of $x^8 - x$ over $\mathbb{Z}_2$ is 3. **6**

(b) Prove that, for each positive divisor m of n, $GF(p^n)$ has a unique subfield of order p^m. Find the number of subfields of $GF(625)$. **6**

OR

(B) (a) Prove that an angle θ is constructible if and only if $\cos\theta$ is constructible. **8**

(b) Prove that a 40º angle is not constructible. **4**

13. (A) (a) Find the Galois group of $\mathbb{Q}(\sqrt[4]{2}, i)$ over $\mathbb{Q}$. **6**

(b) Prove that $\Phi_{2n}(x) = \Phi_n(-x)$ for all odd positive n. **6**

OR

(B) (a) Let N be a normal subgroup of a group G. If both N and G/N are solvable, prove that G is solvable. **6**

(b) Prove that $\Phi_n(x) \in \mathbb{Z}[x]$. **6**

(5 × 12 = 60 Marks)

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Reg. No. : ....

Name : ....

Second Semester M.Sc. Degree Examination, September 2022

Mathematics

MM 224 — PARTIAL DIFFERENTIAL EQUATIONS AND INTEGRAL EQUATIONS

(2020 Admission Onwards)

Time : 3 Hours Max. Marks : 75

PART – A

Answer any **five** questions. Each question carries **3** marks.

1. Solve the PDE $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0$.

2. Solve by Lagrange's method $\left(\frac{y^2 z}{x}\right)\frac{\partial z}{\partial x}+xz\frac{\partial z}{\partial y}=y^2$.

3. Classify the given PDE $x^2u_{xx}-2xyu_{xy}+y^2u_{yy}+xu_x+yu_y=0$.

4. Show that the derivative u_x of a solution $u(x,y)$ to wave equation will also be a solution.

5. Find the eigen values of the Integral Equation $y(s)=\lambda\int_0^1 e^{s+t}y(t)\,dt$.

P.T.O.

6. Find the resolvant kernel for the Integral Equation $y(s) = f(s) + \lambda \int_0^1 e^{s-t} y(t) dt$.

7. Show that extremals of the arc length functionals are straight lines.

8. State Hamilton's principle.

(5 × 3 = 15 Marks)

PART – B

Answer **all** questions. Each question carries **12** marks.

9. (A) (a) Solve the partial differential equation $u_x + u_y = 2$ with the initial condition $u(x,0) = x^2$. **9**

(b) State the generalized Transversality condition. **3**

OR

(B) (a) Find the equation of the surface satisfying the PDE $4yu \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + 2y = 0$ and passing through $y^2 + u^2 = 1$, $x + u = 2$. **6**

(b) Solve the PDE $u_x + 3y^{\frac{2}{3}} u_y = 2$ subject to the initial condition $u(x,1) = 1 + x$. **6**

10. (A) (a) Write the d-Alembert's solution to the wave equation $u_{tt} = c^2 u_{xx}$, $u(x,0) = 0$, $u_t(x,0) = \cos x$. **6**

(b) Reduce $u_{xx} = x^2 u_{yy}$ to canonical form. **6**

OR

(B) (a) Solve the initial value problem $u_x + 2u_y = 0$, $u(0,y) = 4e^{-2x}$ using the method of separation of variables. **6**

(b) Sketch the regions in which the PDE $yu_{xx} - 2u_{xy} + xu_{yy} = 0$ is elliptic, parabolic and hyperbolic. **6**

11. (A) Establish the law of conservation of energy of the wave equation that represents the motion of an infinite string. **12**

OR

(B) Solve the diffusion equation $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ with the initial condition $u(x,0) = e^{-x}$ using the method of Green's function. **12**

12. (A) Find the resolvent kernel of the Integral Equation $y(s) = f(s) + \lambda\int_0^1 (s+t)g(t)\,dt$. **12**

OR

(B) Solve the Integral Equation $y(s) = s + \lambda\int_0^1 \left[st + (st)^{\frac{1}{2}}\right] y(t)\,dt$. **12**

13. (A) Extremize the functional $\mathcal{J}[y(x)] = \int_0^{\frac{\pi}{2}} \left[(y')^2 - y^2\right]dx; y(0) = 0,\ y\left(\frac{\pi}{2}\right) = 1$. **12**

OR

(B) Find the minimal surface of the functional $\mathcal{J}[y(x)] = 2\pi\int_{x_2}^{x_1} y\sqrt{1+(y')}\,dx$. **12**

(5 × 12 = 60 Marks)

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Reg. No. : ....

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Second Semester M.Sc. Degree Examination, September 2022

Mathematics

MM 222 — REAL ANALYSIS II

(2020 Admission Onwards)

Time : 3 Hours Max. Marks : 75

PART – A

Answer **any five** questions. Each question carries **3** marks.

1. Define Lebesgue outer measure and prove that it is countably subadditive and translation invariant.

2. Let $f = g$ a.e. where f is a continuous function. Show that ess sup f =ess sup g = sup f.

3. Show that $\int_0^1 \sin x \log x dx = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)(2n)!}$.

4. Show that the derivatives of a continuous function are measurable.

5. Prove that the limit of a pointwise convergent sequence of measurable functions is measurable.

6. Show that if $0 < a < \infty$ and $0 < p < \infty$ then $\log x^{-1} \in L^p(0, a)$.

7. State and prove Jensen's inequality.

8. Show that if $f_n \to f$ in measure and α is any real number, then $\alpha f_n \to \alpha f$ in measure.

(5 × 3 = 15 Marks)

P.T.O.

PART – B

Answer **any** questions choosing either (a) or (b). Each question carries **12** marks.

9. (A) (a) Prove that the interval (a, ∞) is measurable. **3**

(b) Prove that the Lebesgue outer measure of an interval is its length. **9**

OR

(B) (a) Let $< E_i >$ be a sequence of measurable sets. Prove that $m\left(\bigcup E_i\right) \leq \sum mE_i$. If the sets E_i are pairwise disjoint, then prove that $m\left(\bigcup E_i\right) = \sum mE_i$. **6**

(b) Give an example of a measurable set that is not a Borel set. **6**

10. (A) Prove that if *f* is Riemann integrable and bounded over the finite interval $[a, b]$, then f is integrable and $R\int_a^b f\, dx = \int_a^b f\, dx$. What can you say of the converse? Justify. **12**

OR

(B) (a) Prove that if $f \in L(a, b)$ then $F(x) = \int_a^x f(t)\, dt$ is a continuous function on $[a, b]$ and is of bounded variation on $[a, b]$. **6**

(b) If *f* is a finite-valued monotone increasing function defined on the finite interval $[a, b]$, then prove that f' is measurable and $\int_a^b f'dx \leq f(b) - f(a)$. **6**

11. (A) Prove that if μ is a σ-finite measure on a ring $\mathbb{R}$, then it has a unique extension to the σ-ring $S(\mathbb{R})$. **12**

OR

(B) If μ is a measure on a σ-ring S, then prove that the class $\overline{S}$ of sets of the form $E \Delta N$ for any sets E, N such that $E \in S$ while N is contained in some set in S of zero measure, is a σ-ring and the set function $\overline{\mu}$ defined by $\overline{\mu}(E \Delta N) = \mu(E)$ is a complete measure on $\overline{S}$ **12**

12. (A) (a) Prove that every function that is convex on an open interval is continuous. **6**

(b) State and prove Minkowski's inequality. Also discuss when equality occurs. **6**

OR

(B) Prove that for $p \geq 1$, $L^p(\mu)$ is a complete metric space. **12**

13. (A) Prove that the signed measure on [[X, S]] has a Jordan decomposition. Show also that this decomposition is unique and minimal. **12**

OR

(B) State and prove the Radon-Nikodym theorem. **12**

(5 × 12 = 60 Marks)

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Reg. No. : ....

Name : ....

Second Semester M.Sc. Degree Examination, September 2022

Mathematics

MM 223 – TOPOLOGY II

(2020 Admission onwards)

Time : 3 Hours Max. Marks : 75

PART – A

Answer **any five** questions. **Each** question carries **3** marks.

1. Prove or disprove : countable product of second countable spaces is second countable.

2. Prove that the projection maps $p_i : X \to X_i$, where $X = X_1 \times X_2 \times .... \times X_n$ are continuous.

3. Show that $R / \sim$ is topologically equivalent to a circle.

4. Define T_i-spaces for i = 1, 2 and give an example for a T_1-space which is not T_2.

5. Prove or disprove : product of any family of regular spaces need not be regular.

6. If $f : X \to Y$ then show that f is continuous at $x_0 \in X$ if and only if whenever $\mathscr{F} \to x_0$ in X then $f(\mathscr{F}) \to f(x_0)$ in Y.

7. Prove or disprove : every contractible space is simply connected.

8. Is the set of end points $E = \{a, b\}$ a retract of a closed interval $[a, b]$ where $a < b$? Justify your answer.

(5 × 3 = 15 Marks)

P.T.O.

PART – B

Answer **all** questions. Each question carries **12** marks.

9. A. (a) Prove that product of an arbitrary Collection of connected spaces is connected. **6**

(b) Define (i) Weak topology (ii) Projection map (iii) Quotient space. **6**

OR

B. (a) Prove that product of a finite number of compact spaces is compact. **6**

(b) Let X and Y be spaces and $f : X \to Y$ be a continuous function from X onto Y. Prove that the natural correspondence $h : X / \sim \to Y$ defined by $h([x]) = f(x),\ x \in X$ is a homeomorphism if and only if Y has the quotient topology determined by f. **6**

10. A. State and prove Tietze extension theorem. **12**

OR

B. (a) Show that every metric space is normal. **6**

(b) Prove that Sorgenfrey plane is regular but not normal. **6**

11. A. State and prove Tychonoff theorem; prove at least one significant result used in it. **12**

OR

B. (a) Show that $\mathscr{F}$ has x as a cluster point if and only if there is a filter $\mathscr{G}$ finer than $\mathscr{F}$ which converges to x. **6**

(b) If X is a first countable space and $E \subset X$, then show that $x \in \overline{E}$ if and only if there is a sequence (x_n) contained in E which converges to x. **6**

12. A. (a) Let X be a path connected space and x_0, x_1 points of X. Show that the fundamental groups $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are isomorphic. **6**

(b) State and prove covering homotopy property. **6**

OR

B. (a) Show that the homotopy class [c], where c is the constant loop whose only value is x_0, is the identity element for $\pi_1(X, x_0)$. **6**

(b) Prove that the fundamental group $\pi_1(S^1)$ is isomorphic to the additive group $\mathbb{Z}$ of integers. **6**

13. A. (a) If D is a deformation retract of a space X and x_0 is a point of D, show that $\pi_1(X, x_0)$ and $\pi_1(D, x_0)$ are isomorphic. **6**

(b) State and prove Brouwer fixed point theorem. **6**

OR

B. Show that the n-sphere S^n is simply connected for $n \geq 2$. **12**

(5 × 12 = 60 Marks)
